RESEARCH LECTURE: PRIME IDEALS IN NOETHERIAN RINGS

SYLVIA WIEGAND
TRIBHUVAN UNIVERSITY

Abstract. We discuss some questions and some results concerning partially ordered sets of prime ideals in Noetherian rings. Our focus is two-dimensional integral domains of polynomials and power series.

This research lecture involves questions that Roger Wiegand, Sylvia Wiegand and many others have been working on for over sixty years. The concepts may seem simple but the material is difficult. In the attempt to make this topic clear and interesting to all levels of readers, we may repeat some material from the preparatory lecture, and we may assume other concepts and results.

resintback

1. Introduction and background

Let R be a commutative Noetherian ring, and let $\mathrm{Spec}(R)$ denote the prime spectrum—the set of prime ideals of R as a partially ordered set, or *poset*, under inclusion. Our work on this topic is motivated by the following question raised by Kaplansky about 1950:

KapQ **Question 1.1.** Which partially ordered sets occur as $\mathrm{Spec}(R)$ for some Noetherian ring R?

This difficult question remains unanswered, although there have been many related results, such as: (1) Hochster's characterization of the prime spectrum of a commutative ring as a topological space [**?**]; (2) Lewis' result that every finite poset is the prime spectrum of a commutative ring [**?**]; (3) The identification of some properties of prime spectra of Noetherian rings [**?**]; (4) Examples of Noetherian rings that do not have certain other properties [**?**],[**?**],[**?**]; (5) Characterizations of prime spectra of specific classes of Noetherian rings or of particular Noetherian rings [**?**],[**?**]. This work is discussed in [**?**], along with other results and references.

Theorem **??** is a generalization of examples due to Nagata, Heitmann and McAdam: prime ideals in a Noetherian ring can exhibit any finite amount of "misbehavior".

swint **Theorem 1.2.** [**?**, Theorem 1] *Let F be an arbitrary finite poset. There exist a Noetherian ring R and a saturated order-embedding $\varphi : F \to \mathrm{Spec}(R)$ such that φ preserves minimal upper bound sets and maximal lower bound sets. In detail, for $u, v \in F$, we have*

(i) $u < v$ *if and only if* $\varphi(u) < \varphi(v)$;
(ii) v *covers* u *if and only if* $\varphi(v)$ *covers* $\varphi(u)$;
(iii) $\varphi(\mathrm{mub}_F(u, v)) = \mathrm{mub}(\varphi(u), \varphi(v))$; *and*

Date: 22 April, 2015.

(iv) $\varphi(\mathrm{Mlb}_F(u,v)) = \mathrm{Mlb}(\varphi(u), \varphi(v))$.

Here v *covers* u means $v > u$ and no element is between them; the *minimal upper bound set* of u and v is $\mathrm{mub}_F(u,v) = \min(\{w \in F \mid w \geq u, w \geq v\})$; and the *maximal lower bound set* of u and v is $\mathrm{Mlb}_F(u,v) = \max(\{w \in F \mid w \leq u, w \leq v\})$.

Theorem **??**, due to Steve McAdam, guarantees that noncatenary misbehavior cannot be too widespread in the prime spectrum of a Noetherian ring:

McAdam **Theorem 1.3.** [**?**] *Let P be a prime ideal of height n in a Noetherian ring. Then all but finitely many covers of P have height $n+1$.*

One might conjecture from Theorem **??** that, if a finite "bad" subset is removed, the remaining prime ideals of Noetherian rings behave well, like those in excellent rings.[1]

In view of the difficulties in answering Question **??**, many researchers have adjusted the question to characterizing the prime spectra of specific rings or classes of rings. For a one dimensional ring, the spectrum is somewhat trivial, at least in the countable integral domain case. The next step is to consider:

Kapqdim2 **Question 1.4.** Which partially ordered sets occur as $\mathrm{Spec}(R)$ for some Noetherian domain R of dimension two?

Corollary **??** follows from Theorem **??** and is related to Question **??**.

Ledef **Notation 1.5.** For u a height-two element of a two-dimensional poset U, we set

$$L_e(u) := \{v \in U \mid v < u \text{ and } v \text{ is not less than any other element}\}.$$

swintc **Corollary 1.6.** [**?**, Theorem 2] *Let U be a countable poset of dimension two. Assume that U has a unique minimal element and that the set* $\max(U)$ *of maximal elements is finite. Then $U \cong \mathrm{Spec}(R)$ for some countable Noetherian domain R if and only if, for each element u with* $\mathrm{ht}(u) = 2$*, the set $L_e(u)$ is infinite.*

The general answer for two-dimensional domains is still too difficult. Our current and recent investigations, described in the sequel, are aimed at this question:

Question 1.7. Which partially ordered sets occur as $\mathrm{Spec}(R)$ for a Noetherian domain R that is a two-dimensional domain of polynomials or power series?

We show in Section **??** that certain finite subsets help determine prime spectra for power series rings; see Theorems **??** and **??**.

ressppolypow 2. Prime spectra of 2-dim polynomial & power series domains

Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let x, y and z be indeterminates over any of these rings and fields.

pspropsq **Questions 2.1.** Which pairs of the following prime spectra are order-isomorphic? What properties do the prime spectra have? Which, if any, are shown in the diagrams below?

[1]For the definition of "excellent ring" see [**?**, p. 260]. Basically "excellence" means the ring is catenary and has other nice properties that homomorphic images of polynomial rings over a field possess.

(1) Spec($\mathbb{Z}[y]$) (2) Spec($(\mathbb{Z}/2\mathbb{Z})[y]$) (3) Spec($\mathbb{Z}_{(2)}[y]$)
(4) Spec($\mathbb{Q}[x,y]$) (5) Spec($\overline{\mathbb{Q}}[x,y]$) (6) Spec($\mathbb{C}[x,y]$)
(7) Spec($\mathbb{Z}[[x]]$) (8) Spec($\mathbb{Q}[y][[x]]$) (9) Spec($\mathbb{Q}[[x]][y]$)
(10) Spec($\mathbb{C}[x,y,z]/(x^4+y^4+z^4-1)$) (11) Spec($\mathbb{C}[[x,y,z]]/(x^4+y^4+z^4-1)$)
(12) Spec($\mathbb{Z}[[x]][y]/(x-2y^2-2)$), or (13) Spec($\mathbb{Z}[y][[x]]/(x-2y^2-2)$).

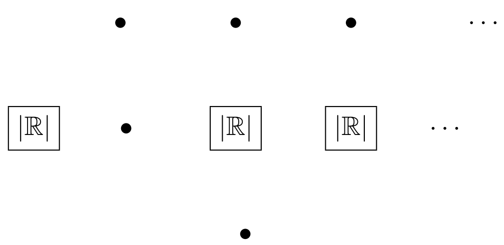

Diagram **??**.a

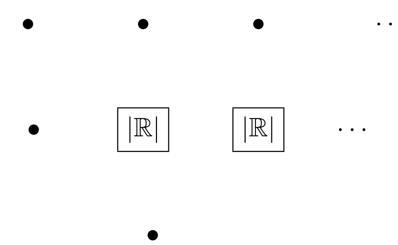

Diagram **??**.b

In each diagram there is a unique height-one element with more than one cover; $|L_e(u)| = |\mathbb{R}|$ for every height-two u; and the number of height-two elements is $\aleph_0 = |\mathbb{Z}|$.

Remarks 2.2. Some answers: (1) The spectra of the rings in #s 1, 2, 3, 4, and 5 are countable and thus are not order-isomorphic to the others, which are uncountable.

(2) Diagram **??**.b shows Spec($R[[x]]$), for every one-dimensional countable Noetherian domain R with $\aleph_0$ maximal ideals [**?**]. Thus Spec($\mathbb{Z}[[x]]$) $\cong$ Spec($\mathbb{Q}[y][[x]]$). Diagram **??**.a shows Spec($\mathbb{Q}[[x]][y]$), and so Spec($\mathbb{Q}[[x]][y]$) $\not\cong$ Spec($\mathbb{Q}[y][[x]]$); see [**?**].

(3) Roger Wiegand gives five axioms that characterize Spec($\mathbb{Z}[y]$) in [**?**]. His axioms imply, for F a field, Spec($\mathbb{Z}[y]$) $\cong$ Spec($F[x,y]$ $\iff$ F is contained in the algebraic closure of a finite field [**?**]. Thus Spec($\mathbb{Z}[y]$) $\cong$ Spec($(\mathbb{Z}/2\mathbb{Z})[x,y]$), but Spec($\mathbb{Z}[y]$) $\not\cong$ Spec($\mathbb{Q}[x,y]$). Translated to algebraic geometry, the crucial axiom for Spec($\mathbb{Z}[y]$) is "For every curve C and finite set of points $P_1,\dots,P_n$ on C, there exists an irreducible curve that meets C in exactly the points $P_1,\dots,P_n$." In $\mathbb{Q}[x,y]$, the point $P=(1,1)$ is on the curve C of $x^3-y^2-1=0$, but every curve that meets C at P meets C in a second point. In ring theory language: The maximal ideal $M=(x-1,y-1)\mathbb{Q}[x,y]$ contains the prime ideal $P=(x^3-y^2-1)\mathbb{Q}[x,y]$. But every height-one prime ideal $Q\subseteq M$ with $Q\neq P$ is such that $P+Q$ is contained in a maximal ideal N of Spec($\mathbb{Q}[x,y]$) with $N\neq M$.

So far no one has characterized Spec($\mathbb{Q}[x,y]$), Spec($\overline{\mathbb{Q}}[x,y]$) or Spec($\mathbb{C}[x,y]$). It is not known if Spec($\mathbb{Q}[x,y]$) $\cong$ Spec($\overline{\mathbb{Q}}[x,y]$) or if Spec($\mathbb{Q}[x,y]$) $\cong$ Spec($\mathbb{Q}[x,y]$) $\setminus\{u\}$, for $u\neq(0)$.

(4) The prime spectrum in #3, Spec($\mathbb{Z}_{(2)}[y]$ (which is countable) is characterized in [**?**]. Its shape is similar to that of Diagram **??**.a *except* that there are countably infinitely many height-ones below each set of n height-two elements, for every $n\in\mathbb{N}$.

(5) The prime spectrum of #10 might appear to be the same as Spec($\mathbb{C}[x,y]$). But in Spec($\mathbb{C}[x,y]$) every maximal ideal is uniquely determined by a pair of height-one prime ideals; e.g. the maximal ideal $(x-2,y-3)\mathbb{C}[x,y]$ is unique for $(x-2)$ and $(y-3)$. In algebraic geometric terms "Every point is the set-theoretic intersection of two curves". Roger Wiegand has shown this property fails for spectrum # 10; see [**?**, Cor 3].

(6) The spectrum of # 11, given in [**?**], is not like the others because the ring is local.

(7) We discuss spectra # 12 and # 13 briefly in Subsection **??**.

3. Prime spectra for quotients of mixed polynomial-power series rings

prqpolpow

We give a sample of results from joint work with Ela Celibas and Christina Eubanks-Turner on prime spectra of quotients of mixed polynomial-power series rings over a Noetherian one-dimensional domain R or a field k. For example, we show:

ppsthm1 **Theorem 3.1.** [**?**] *Let R be a countable Noetherian one-dimensional domain with infinitely many maximal ideals, let x and y be indeterminates over R. Let Q be a height-one prime ideal of $R[y][[x]]$ such that $x \notin Q$ and $\dim(R[y][[x]]/Q) = 2$. Then there exists a finite partially ordered set F of dimension at most one and an order-embedding $\varphi : F \hookrightarrow U$ such that $U = \operatorname{Spec}(R[y][[x]]/Q)$ is determined by $\varphi(F)$ and the following properties:*

(0) U is a two-dimensional partially ordered set with a unique minimal element.

(1) $\operatorname{ht}(\varphi(f)) = 1 + \operatorname{ht}(f)$, for every $f \in F$, and φ preserves minimal upper bounds,

that is, for every pair $f, g \in F$, $\varphi(\operatorname{mub}_F(f, g)) = \operatorname{mub}_U(\varphi(f), \varphi(g))$.

(2) For every $u \in U$ of height two, (i) there exists an $f \in \min F$ such that $\varphi(f) < u$,

and (ii) $|L_e(u)| = |\mathbb{R}|$ ($|L_e(u)|$ as in Corollary **??***).*

(3) For every $u \in \min \varphi(F)$, $|\{w \in U \mid u < w\}| = \aleph_0$.

(4) If $u \in U \setminus \varphi(F)$ and $\operatorname{ht} u = 1$, then $|\{w \in U \mid u < w\}| = 1$.

Diagram **??**.0 of $\operatorname{Spec}(\mathbb{Z}[y][[x]]/(x-2y^2+2))$ illustrates Theorem **??**, except that we cannot show the boxes of cardinality $|\mathbb{R}|$ below *every* height-two element:

$\boxed{\aleph_0; \overline{(x, y-1, 5)} \in}$ $\quad \overline{(x, 2, y-1)}$ $\quad \boxed{\aleph_0}$ $\quad \overline{(x, 2, y+1)}$ $\quad \boxed{\aleph_0; \overline{(x, y+1, 3)} \in}$

$\overline{(x, y-1)}$ $\quad \boxed{|\mathbb{R}|}$ $\quad \overline{(x, 2)}$ $\quad \boxed{|\mathbb{R}|}$ $\quad \overline{(x, y+1)}$

$(\bar{0}) = \overline{(x - 2(y-1)(y+1))}$

Diagram **??**.0: $\operatorname{Spec}(\mathbb{Z}[y][[x]]/(x-2(y-1)(y+1))$.

In Diagram **??**.0, the partially ordered set F of Theorem **??** can be taken to be the finite subset $\{(x, y-1), (x, 2), (x, y+1), (x, y-1, 2), (x, y+1, 2)\}$ in $\operatorname{Spec}(\mathbb{Z}[y][[x]])$. Here "$\overline{}$" denotes "image in $\mathbb{Z}[y][[x]]/(x-2(y-1)(y+1))$".

ppsthm2 **Theorem 3.2.** [**?**] *Let E be a finite partially ordered set of dimension at most one. Let $F = \min E \cup \bigcup_{u \neq v, u,v \in \min E} \operatorname{mub}(u, v)$. Then there exists a height-one prime ideal Q of $\mathbb{Z}[y][[x]]$ and an embedding $\varphi : F \hookrightarrow U$, such that $U = \operatorname{Spec}(\mathbb{Z}[y][[x]]/Q)$ satisfies properties 0 - 4 of Theorem* **??** *for F and φ.*

Results similar to Theorems **??** and **??** hold for two-dimensional prime spectra $U' = \operatorname{Spec}(R[[x]][y]/Q)$ over a countable Noetherian one-dimensional domain R

with infinitely many maximal ideals, where $Q \neq (x)$ is a height-one prime ideal of $R[[x]][y]$, *except* that there may be height-one maximal elements in U'. Also we have results concerning prime spectra for two-dimensional images of mixed polynomial-power series over semilocal Noetherian one-dimensional domains and for two-dimensional images $k[[x]][y,z]/Q$ of polynomial-power series over a field k, where again $Q \neq (x)$ and Q is a height-one prime ideal of $k[[x]][y,z]$, and for rings or fields that are uncountable. The prime spectra of one-dimensional domains of course are rather trivial; we provide conditions that yield the one-dimensional case, and we give the cardinalities that occur in that case [**?**].

References

[1] Ela Celikbas, Christina Eubanks-Turner, and Sylvia Wiegand *Prime ideals in quotients of mixed polynomial-power series rings*, in progress.

[2] William Heinzer, Christel Rotthaus, and Sylvia Wiegand, *Mixed polynomial-power series rings and relations among their spectra*, Multiplicative ideal theory in commutative algebra, Springer, New York, 2006, pp. 227–242.

[3] William Heinzer and Sylvia Wiegand, *Prime ideals in two-dimensional polynomial rings*, Proc. Amer. Math. Soc. **107** (1989), no. 3, 577–586.

[4] Raymond C. Heitmann, *Prime ideal posets in Noetherian rings*, Rocky Mountain J. Math. **7** (1977), no. 4, 667–673.

[5] Melvin Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **142** (1969), 43–60.

[6] William Lewis, *The spectrum of a ring as a partially ordered set*, J. Algebra. **25** (1973), 419–434.

[7] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.

[8] Stephen McAdam, *Saturated chains in Noetherian rings*, Indiana Univ. Math. J. **23** (1973/74), 719–728.

[9] Stephen McAdam, *Intersections of height* 2 *primes*, J. Algebra **49** (1977), no. 2, 315–321.

[10] M. Nagata, On the chain problem of prime ideals, Nagoya Math. J. 10 (1956), 51–64.

[11] L. J. Ratliff, *Chain conjectures in ring theory*, Lecture Notes in Mathematics, vol. 647, Springer, Berlin, 1978, An exposition of conjectures on catenary chains.

[12] Roger Wiegand, *The prime spectrum of a two-dimensional affine domain*, J. Pure Appl. Algebra **40** (1986), no. 2, 209–214.

[13] Roger Wiegand and Sylvia Wiegand, *Prime ideals in Noetherian rings: A survey* in Ring and module theory 175, Trends Math., Birkhouser/Springer Basel AG, Basel, 2010.

[14] Sylvia Wiegand and Roger Wiegand, *The maximal ideal space of a Noetherian ring*, J. Pure Appl. Algebra **08** (1976), 129–141.

[15] Sylvia Wiegand, *Intersections of prime ideals in Noetherian rings*, Comm. Algebra 11 (1983), 1853–1876.