GOING-UP IMPLIES GENERALIZED GOING-UP

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Abstract. Let $R \subseteq D$ be a going-up extension of rings. The purpose of this paper is to show that $R \subseteq D$ is a generalized going-up extension.

In this paper all rings are commutative rings, and with 1 if necessary. Our aim is to show the following.

(1) Let $(P_\alpha)_{\alpha\in\Omega}$ be a chain of prime ideals of an integral domain R. Then there exists a valuation domain V containing R with a chain of prime ideals $(Q_\alpha)_{\alpha\in\Omega}$ lying over $(P_\alpha)_{\alpha\in\Omega}$.
(2) Let $D \supseteq R$ be a going-up ring extension. Then it is a generalized going-up extension, in other words, for any chain $P_0 \subseteq (P_\alpha)_{\alpha\in\Omega}$ of prime ideals of R and a prime ideal Q_0 of D lying over P_0, there exists a chain of prime ideals $Q_0 \subseteq (Q_\alpha)_{\alpha\in\Omega}$ lying over $(P_\alpha)_{\alpha\in\Omega}$.

We will try to make this survey paper as self-contained as possible.

Definition 1. A ring V is a valuation ring if for every two elements a, b of V, $a \mid b$ or $b \mid a$.

Remark 2. V is a valuation domain if and only if for every element x in the quotient field of V, x or $1/x$ is in V. In other words, $K = V \cup (V \setminus 0)^{-1}$, where K is the quotient field of V. Also V is a valuation domain if and only if all the ideals of V are linearly ordered.

Theorem 3. *Let $V \supseteq W$ be valuation domains with the same quotient field. Then $V = W_P$ for some prime ideal P of W.*

Proof. Let M be the maximal ideal of V and $P := M \cap W$. We claim that $V = W_P$. Clearly $W_P \subseteq V$. Conversely let $x \in V$. Since W is a valuation domain, x or $1/x$ is in W. Suppose that x is not in W; then $1/x$ is in W. We show that $1/x$ is not in P. For otherwise $1/x \in P \subseteq M$, which implies that M contains a unit element $1/x$, a contradiction. Thus $1/x \in W \setminus P$ and hence $x = 1/(1/x) \in W_P$. □

Corollary 4. *Let $V \supseteq W$ be valuation domains with the same quotient field. Then* $\mathrm{Spec}(V) \subseteq \mathrm{Spec}(W)$.

Proof. Let M be the maximal ideal of V and let $P := M \cap W$. We have $V = W_P$ by Theorem **??** and $M = P_P$. However $P_P = P$. For $s \in W \setminus P$, $P \subseteq sW$ and hence $P \subseteq sP$. Therefore $s^{-1}P \subseteq P$. Thus $M = P_P = P$, which implies that every prime ideal of V is also a prime ideal of W. □

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Let $D \supseteq R$ be a ring extension of R. An element x of D is integral over R if $f(x) = 0$ for some monic polynomial f over R, i.e.,

$$x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0 = 0$$

for some elements $r_0, r_1, \ldots, r_{n-1}$ of R.

Thus x being integral over R means that $x(1, x, x^2, \ldots, x^n) \subseteq (1, x, x^2, \ldots, x^n)$ for some $n \geq 1$. Note that $(1, x, x^2, \ldots, x^n)$ is a finitely generated R-module and $\mathrm{Ann}((1, x, x^2, \ldots, x^n)) = \{0\}$.

Theorem 5. *Let R be a ring with 1. Let x be an element in an extension ring D of R. Then x is integral over R if and only if there exists a finitely generated R-module $I \subseteq D$ such that $\mathrm{Ann}(I) = \{0\}$ and $xI \subseteq I$.*

Proof. It is an easy exercise using Cramer's rule. □

Corollary 6. *Integral elements are closed under addition and multiplication.*

Proof. Let $x \in D \supseteq R$ be integral over R. x being integral over R means that there exists a finitely generated R-module $I \subseteq D$ such that $\mathrm{Ann}(I) = \{0\}$ and $xI \subseteq I$. For finitely many integral elements $x_1, \ldots, x_m$ over R, choose finitely generated R-modules $I_1, \ldots, I_m$ as above. Then $(x_1, \ldots, x_m)I_1 \cdots I_m \subseteq I_1 \cdots I_m$ and $I_1 \cdots I_m$ is a finitely generated R-module such that $\mathrm{Ann}(I_1 \cdots I_m) = \{0\}$. □

Theorem 7. *(Lying-over). Let $R' \supseteq R$ be an integral ring extension of R. Let P be a prime ideal of R. Then there exists a prime ideal P' of R' lying over P.*

Proof. Let $S := R \setminus P$. Then S is a multiplicatively closed subset of R and hence of R'. Let Γ be the collection of ideals Q of R' such that $Q \supseteq P$ and $Q \cap S = \emptyset$. First we show that Γ is nonempty.

Claim: $PR' \cap S = \emptyset$.

Note that PR' is integral over P. Indeed look at each element of the form px, where $p \in P$ and $x \in R'$. Since x is integral over R,

$$x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0 = 0$$

for some elements $r_0, r_1, \ldots, r_{n-1}$ of R. From this equation, we get

$$(px)^n + r_{n-1}p(px)^{n-1} + \cdots + r_1 p^{n-1}(px) + r_0 p^n = 0,$$

an integral equation for px over P, and hence any finite combination of them is integral over P by Corollary **??**. Now suppose $s = x$, where $s \in S$ and $x \in PR'$. Then $s = x$ is integral over P. This is a contradiction since

$$s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0 = 0,$$

where each $p_i \in P$, implies $s^n \in P$.

Thus $\Gamma \neq \emptyset$. Choose a maximal element of Γ, i.e., an ideal Q maximal with respect to the property: $Q \supseteq P$ and $Q \cap S = \emptyset$. We know by Krull (see [**?**, Theorem 1]) that Q is a prime ideal. Clearly Q lies over P. □

Theorem 8. *Let R be an integral domain and P be a prime ideal of R. Then there is a valuation overring (V, M) of R such that M contracts to P.*

Proof. Since $P = PR_P \cap R$, it suffices to prove the theorem under the assumption that R is quasi-local with maximal ideal P.

Let $\mathcal{S}$ be the set of overrings (A, H) of R such that H is a proper ideal of A and $H \supseteq P$. We order elements of $\mathcal{S}$ by $(A, H) \geq (B, G)$ if and only if $A \supseteq B$ and $H \supseteq G$.

Let (A_α, H_α) be a chain of elements in $\mathcal{S}$. Then $H := \cup_\alpha H_\alpha$ is a proper ideal of the ring $A := \cup_\alpha A_\alpha$. So (A, H) is an upper bound. By Zorn's lemma, $\mathcal{S}$ has a maximal element, say (A, H).

Claim: A is a valuation domain.

Let A' be the integral closure of A. Since $A' \supseteq A$ is a LO extension, $A' = A$ by the maximality of A. Thus A is integrally closed.

Clearly A is a quasi-local ring with maximal ideal H since $(A_M, H_M) \geq (A, H)$ and hence $(A_M, H_M) = (A, H)$ by the maximality of (A, H), where M is a prime ideal containing H. Let y be a nonzero element of K, the quotient field of both R and A. We show that y or $1/y$ is in A. First note that y or $1/y$ is in W since W is a valuation domain. Let x be y or $1/y$, which is contained in W. Thus $x \in W$.

Case 1: The unity $1 \in HA[x] = H[x]$.

Then $1 = h_0 + h_1 x + \cdots + h_n x^n$ and hence $h_1 x + \cdots + h_n x^n = 1 - h_0$ is a unit in A since A is a quasi-local ring. So $1/x$ is integral over A and hence $1/x$ is in A since A is integrally closed. Hence y or $1/y$ is in A.

Case 2: The unity $1 \notin HA[x] = H[x]$.

Note that $A[x] \subseteq W$ since $x \in W$. Now $(A[x], H[x]) \geq (A, H)$ in $\mathcal{S}$ and hence $A[x] = A$ by the maximality of (A, H), whence $x \in A$. Thus x or $1/x$ is in A and hence y or $1/y$ is in A.

Thus in either case, y or $1/y$ is in A so that A is a valuation domain. Now by the definition of $\mathcal{S}$, $P \subseteq H \cap R$, and the reverse containment is obvious since R is quasi-local. Hence $P = H \cap R$. □

Theorem 9. *(1) Let $P_1 \subseteq \cdots \subseteq P_n$ be a chain of prime ideals of an integral domain R. Then there is a valuation overring V of R with prime ideals $Q_1 \subseteq \cdots \subseteq Q_n$ lying over $P_1 \subseteq \cdots \subseteq P_n$, where Q_n is the maximal ideal of V.*

(2) Moreover, if $P_1 \subseteq \cdots \subseteq P_n \subseteq P_{n+1}$ is a chain of prime ideals of an integral domain R and W is a valuation overring of R with prime ideals $Q_1 \subseteq \cdots \subseteq Q_n$ lying over $P_1 \subseteq \cdots \subseteq P_n$, where Q_n is the maximal ideal of W, then there exists a valuation overring $V \subseteq W$ of R with prime ideals $Q_1 \subseteq \cdots \subseteq Q_n \subseteq Q_{n+1}$ lying over $P_1 \subseteq \cdots \subseteq P_n \subseteq P_{n+1}$, where Q_{n+1} is the maximal ideal of V.

Proof. We prove the theorem by induction on n. If P_1 is a prime ideal of R, then by Theorem **??** there exists a valuation overring (W, Q_1) of R such that Q_1 lies over P_1. Hence, (1) is true for $n = 1$.

Suppose that (1) is true for n. Let $P_1 \subseteq \cdots \subseteq P_n \subseteq P_{n+1}$ be a chain of prime ideals of R and let $W \supseteq R$ be a valuation overring of R with prime ideals $Q_1 \subseteq \cdots \subseteq Q_n$ lying over $P_1 \subseteq \cdots \subseteq P_n$, where Q_n is the maximal ideal of W.

By Theorem **??** with $(R/P_n, P_{n+1}/P_n)$, there exists a valuation overring (V', Q') of R/P_n such that $Q' \cap R/P_n = P_{n+1}/P_n$. Since W/Q_n is the quotient field of R/P_n, $V' \subseteq W/Q_n$. Let V be the inverse image of V' under the canonical epimorphism $W \to W/Q_n$. Then V is a valuation domain. Obviously, $R \subseteq V \subseteq W$. Let Q_{n+1} be the maximal ideal of V. By Corollary **??**, $\mathrm{Spec}(W) \subseteq \mathrm{Spec}(V)$ so that

$Q_1 \subseteq \cdots \subseteq Q_n \subseteq Q_{n+1}$ is a chain of prime ideals of V. Finally, the equality $Q' \cap R/P_n = P_{n+1}/P_n$ implies $Q_{n+1} \cap R = P_{n+1}$. This proves (1) and also (2). □

Theorem 10. *Let $P_1 \subseteq \cdots \subseteq P_n \subseteq \cdots$ be an ascending chain of prime ideals of an integral domain R. Then there exists a valuation domain $V \supseteq R$ with a chain of prime ideals $Q_1 \subseteq \cdots \subseteq Q_n \subseteq \cdots$ lying over $P_1 \subseteq \cdots \subseteq P_n \subseteq \cdots$.*

Proof. Using Theorem **??**, we successively find a decreasing sequence of valuation domains V_n and a sequence of prime ideals $Q_1 \subseteq \cdots \subseteq Q_n$ of V_n, which lies over $P_1 \subseteq \cdots \subseteq P_n$. Put $V = \cap_{n=1}^{\infty} V_n$. Then V is a valuation domain and $Q_1 \subseteq \cdots \subseteq Q_n \subseteq \cdots$ is contained in V by Corollary **??**. □

If the chain $(P_\alpha)_{\alpha\in\Omega}$ is not countable, then we cannot apply the inductive method. We will present a method which does not appeal to the inductive method.

Let $(P_\alpha)_{\alpha\in\Omega}$ be a chain of prime ideals of an integral domain R. We will find a valuation domain V with a chain of prime ideals $(Q_\alpha)_{\alpha\in\Omega}$ lying over $(P_\alpha)_{\alpha\in\Omega}$.

Let

$$S = \bigcup_{\alpha\in\Omega} (R \setminus P_\alpha)^{-1} P_\alpha.$$

Lemma 11. *S is closed under multiplication and multiplication by elements of R.*

Proof. It is an easy exercise. □

Lemma 12. *$\langle S\rangle := SR[S]$ is a proper ideal of $R[S]$.*

Proof. Suppose not. Then $1 = s_1 + \cdots + s_n$ by Lemma **??**, where each $s_i \in (R\setminus P_i)^{-1}P_i$. Since $(P_\alpha)_{\alpha\in\Omega}$ is a chain, we may assume that $P_n \subseteq \cdots \subseteq P_1$. Choose a valuation domain V with prime ideals $Q_n \subseteq \cdots \subseteq Q_1$ lying over $P_n \subseteq \cdots \subseteq P_1$. Note that each $s_i \in (R \setminus P_i)^{-1}P_i \subseteq P_{i P_i} \subseteq Q_{i Q_i} = Q_i$. So $1 = s_1 + \cdots + s_n \in Q_1$, a contradiction. Thus $\langle S\rangle$ is a proper ideal of $R[S]$. □

We can now prove the following theorem, which is the main result in [**?**].

Theorem 13. *Let $(P_\alpha)_{\alpha\in\Omega}$ be a chain of prime ideals of an integral domain R. Then there exists a valuation domain V with a chain of prime ideals $(Q_\alpha)_{\alpha\in\Omega}$ lying over $(P_\alpha)_{\alpha\in\Omega}$.*

Proof. Let P be a prime ideal of $R[S]$ containing $\langle S\rangle$. Choose a valuation domain (V, M) centered on P, i.e., $M \cap R[S] = P$. Let $A_\alpha = \sqrt{P_\alpha V}$, which is a prime ideal of V. We claim that $A_\alpha \cap R = P_\alpha$. Suppose $P_\alpha \subset A_\alpha \cap R$ and choose $f \in (A_\alpha \cap R)\setminus P_\alpha$. Then $f^n \in p_\alpha V$ for some $n \geq 1$ and $p_\alpha \in P_\alpha$. Now $\frac{f^n}{p_\alpha} \in V$, which implies $\frac{p_\alpha}{f^n}$ is a unit, which however is in $(R \setminus P_\alpha)^{-1}P_\alpha \subseteq S \subseteq SR[S] \subseteq P \subseteq M$, a contradiction. Thus $P_\alpha = A_\alpha \cap R$ for each α. □

Definition 14. *(Going-up extension).* Let $D \supseteq R$ be a ring extension. It is a going-up extension if given prime ideals $P_0 \subseteq P_1$ of R and a prime ideal Q_0 of D lying over P_0, we can find a prime ideal $Q_1 \supseteq Q_0$ of D lying over P_1.

Definition 15. *(Generalized going-up extension).* Let $D \supseteq R$ be a ring extension. It is a generalized going-up extension if for any chain $P_0 \subseteq (P_\alpha)_{\alpha\in\Omega}$ of prime ideals of R and a prime ideal Q_0 of D lying over P_0, there exists a chain of prime ideals $(Q_\alpha)_{\alpha\in\Omega} \supseteq Q_0$ of D lying over $(P_\alpha)_{\alpha\in\Omega}$.

Theorem 16. *(Kang-Oh). GU implies GGU.*

Proof. The proof is basically the same as that of Theorem **??**. □

Definition 17. *(Going-down extension).* Let $D \supseteq R$ be a ring extension. It is a going-down extension if given prime ideals $P_1 \subseteq P_0$ of R and a prime ideal Q_0 of D lying over P_0, we can find a prime ideal $Q_1 \subseteq Q_0$ of D lying over P_1.

Definition 18. *(Generalized going-down extension).* Let $D \supseteq R$ be a ring extension. It is a generalized going-down extension if for any chain $(P_\alpha)_{\alpha\in\Omega} \subseteq P_0$ of prime ideals of R and a prime ideal Q_0 of D lying over P_0, there exists a chain of prime ideals $(Q_\alpha)_{\alpha\in\Omega} \subseteq Q_0$ of D lying over $(P_\alpha)_{\alpha\in\Omega}$.

Theorem 19. *GD implies GGD.*

Proof. Use Hochster's duality theorem: For a ring R, there is a ring R' such that $\operatorname{Spec}(R')$ is reversely isomorphic to $\operatorname{Spec}(R)$ as posets. □

For more about LO, GU, and GD, we refer the readers to [**?**, **?**].

Theorem 20. *(Kang-Oh). Let $D \supseteq R$ be a going-up extension. Then every tree in $\operatorname{Spec}(R)$ can be embedded in a tree of $\operatorname{Spec}(D)$.*

Proof. An ardous checking of all the cases will do the work. See [**?**]. □

Corollary 21. *Let R be a Prüfer domain and let D be the integral closure of R in any extension field of the quotient field of R. Then $\operatorname{Spec}(R) \hookrightarrow \operatorname{Spec}(D)$.*

Remark 22. In the above case, both $\operatorname{Spec}(R)$ and $\operatorname{Spec}(D)$ are trees, one of which is a subtree of the other.

Question 23. How many embeddings are there?

Theorem 24. *(Kang-Oh). Let R be an integral domain. Then there exists a Bézout domain $D \supseteq R$ such that every tree T in $\operatorname{Spec}(R)$ is a contraction of a tree T' in $\operatorname{Spec}(D)$.*

Proof. Let $D = (R')^b$ be the Kronecker function ring of the integral closure R' of R. It is well-known that D is a Bézout ring. The extensions $R \subseteq R' \subseteq (R')^b$ are going-up extensions, and hence $R \subseteq (R')^b$ is a going-up extension. □

The following is a result in [**?**].

Theorem 25. *(Kang). Every tree can be embedded into the spectrum of a Prüfer domain.*

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