LOCAL RINGS

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In this talk I will introduce local rings (mostly commutative) and, through localization, show how they can be used in commutative algebra.

sec:terminology

1. Terminology

For many people working in commutative algebra, particularly ideal theory of integral domains, a *local ring* is simply a commutative ring with exactly one maximal ideal. (Thus the zero ring is not local.) For people working on rings related to number theory and algebraic geometry, however, local rings are assumed to be Noetherian, and "quasi-local" is used for commutative rings that may not be Noetherian. What about non-commutative rings? There, "local" means that the sum of any two non-units is again a non-unit. In order to avoid a collision of terminology, we will use the term "nc-local" when non-commutativity is allowed, reserving the term "local" for commutative Noetherian rings. To summarize:

def:local-terminology

Definition 1.1. Let R be a ring (always with identity!)

(a) R is *nc-local* provided $R \neq 0$ and the sum of any two non-units is again a non-unit.

(b) R is *quasi-local* provided R is commutative and has exactly one maximal ideal.

(c) R is *local* provided R is quasi-local and Noetherian.

ex:local

Exercise 1.2. Prove that in an nc-local ring R the set J of non-units is exactly the Jacobson radical $J(R)$. Show that a ring A is nc-local if and only if $A/J(A)$ is a division ring. Prove that a commutative ring is quasi-local if and only if it is nc-local.

2. Localization

Let R be an arbitrary commutative ring, and let S be a *multiplicative set*, that is, a subset of R closed under multiplication and containing 1. (Some people omit the requirement that $1 \in S$, and, as it turns out, this doesn't matter much, as long as $S \neq \emptyset$.) Let M be any R-module (always assumed to be unital, that is, $1x = x$ for each $x \in M$). Define an equivalence relation $\sim$ on $M \times S$ as follows:

$$(x, s) \sim (y, t) \qquad \Longleftrightarrow \exists u \in S \text{ such that } u(tx - sy) = 0$$

(Prove that this is indeed an equivalence relation.) As one does in constructing the rational numbers from the integers, we let $\frac{x}{s}$ denote the equivalence class of the pair (x, s). We make the set $S^{-1}M$ of equivalence classes $\frac{x}{s}$ into an abelian group in the obvious way: $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$. Moreover, $S^{-1}R$ is a commutative ring with

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the multiplication $\frac{a}{s}\frac{b}{t} = \frac{ab}{st}$, and $S^{-1}M$ is an $S^{-1}R$-module: $\frac{a}{s}\frac{x}{t} = \frac{ax}{st}$. (Prove that these operations are well-defined and do indeed provide rings and modules.) These gadgets are called the *ring of fractions of R with respect to S*, and the *module of fractions of M with respect to S*.

Suppose $A \xrightarrow{f} B$ is a homomorphism of R-modules. Show that we get a homomorphism $S^{-1}f : S^{-1}A \to S^{-1}B$ of $S^{-1}R$-modules by sending $\frac{a}{s}$ to $\frac{f(a)}{s}$. Taking modules of fractions is a functor; in particular, if we have R-homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$, then $S^{-1}(gf) = (S^{-1}g)(S^{-1}f)$.

One of the most important examples of a multiplicative set is the set-theoretic complement of a prime ideal. More generally, let $\mathcal{P}$ be any set of prime ideals of R; then $S := R \setminus \bigcup \mathcal{P}$ is a multiplicative set. When P is a prime ideal of R, we usually write R_P and M_P instead of $(R \setminus P)^{-1}R$ and $(R \setminus P)^{-1}M$. For a homomorphism f, we write f_P instead of $(R \setminus P)^{-1}f$. We refer to R_P and M_P as the *localization* of R (respectively M) at P.

3. Exactness

Recall that a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is *exact* provided $\operatorname{im} f = \ker g$. Taking modules of fractions is an exact functor:

ex:exact

Exercise 3.1. Let S be a multiplicative subset of R, and let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of R-modules. Prove that $S^{-1}A \xrightarrow{S^{-1}f} S^{-1}B \xrightarrow{S^{-1}g} S^{-1}C$ is exact.

Example 3.2. Let k be a field and $R = \mathsf{k}[x_1, \dots, x_n]$ the polynomial ring in n variables over k. Let $p \in \mathsf{k}^n$, say, $p = (a_1, \dots, a_n)$, with each $a_i \in \mathsf{k}$. Then $\mathfrak{m}_p := \{f \in R \mid f(p) = 0\}$ is a maximal ideal of R, and $R_\mathfrak{m}$ is the ring of rational functions $f = \frac{r}{s}$, where $r, s \in R$ and $s(p) \neq 0$. Thus $R_\mathfrak{m}$ is the ring of rational functions that are defined at p. (Exercise: Show that the ideal $\mathfrak{m}_p$ is generated by the elements $x_1 - a_1, \dots, x_n - a_n$.)

4. Globalization

The gist of the following theorem is that if a module is zero locally then it is really zero. First of all, when is an element $\frac{x}{s}$ in $S^{-1}M$ equal to zero? Answer: $\frac{x}{s} = \frac{0}{1}$ if and only if there is an element $t \in S$ such that $tx = 0$.

thm:globalization

Theorem 4.1. *Let R be a commutative ring and A an R-module. These are equivalent:*
(a) $A = 0$.
(b) $S^{-1}A = 0$ for every multiplicative set S in R.
(c) $A_P = 0$ for every prime ideal P of R.
(d) $A_\mathfrak{m} = 0$ for every maximal ideal $\mathfrak{m}$ of R.

The only non-trivial implication is that (d) $\implies$ (a). If $A \neq 0$, choose a non-zero element x of A, and let $I = \{r \in R \mid rx = 0\}$, the *annihilator* of x. Then I is a proper ideal of R (since $1 \notin I$), and Zorn's lemma implies that there is a maximal ideal $\mathfrak{m}$ containing I. Thus nothing outside $\mathfrak{m}$ annihilates $\frac{x}{1}$, and so $\frac{x}{1}$ is a non-zero element of $A_\mathfrak{m}$.

cor:exact-global

Corollary 4.2. *Let $A \xrightarrow{f} B$ be a homomorphism of R-modules, where R is a commutative ring.*

(a) f is injective if and only if $f_{\mathfrak{m}}$ is injective for every maximal ideal $\mathfrak{m}$ of R.

(b) f is surjective if and only if $f_{\mathfrak{m}}$ is surjective for every maximal ideal $\mathfrak{m}$ of R.

(c) f is an isomorphism if and only if $f_{\mathfrak{m}}$ is an isomorphism for every maximal ideal $\mathfrak{m}$ of R.

Proof. Notice that f is injective if and only if $0 \to A \xrightarrow{f} B$ is exact. Therefore, if f is injective, so is $f_{\mathfrak{m}}$, by Exercise **??**. For the converse, let $K = \ker f$. Then $0 \to K \xrightarrow{\subseteq} A \xrightarrow{f} B$ is exact, and, by Exercise **??**, so is $0 \to K_{\mathfrak{m}} \xrightarrow{\subseteq} A_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} B_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$. Injectivity of $f_{\mathfrak{m}}$ forces $K_{\mathfrak{m}} = 0$ for each $\mathfrak{m}$, and now Theorem **??** implies that $K = 0$, that is, f is injective. This proves (a). The proof of (b) is similar; use the cokernel C and the exact sequence $A \xrightarrow{f} B \to C \to 0$. Of course (c) follows from (a) and (b). □

Warning: This does *not* say that locally isomorphic modules are actually isomorphic. That is, one can have non-isomorphic modules A and B such that $A_{\mathfrak{m}} \cong B_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$. Without such examples, number theory and algebraic topology would be uttterly boring: The ring $\mathbb{Z}[\sqrt{-5}]$ would be a UFD, Fermat's Last Theorem would have been proved two centuries ago, all vector bundles would be trivial, there would be no Moebius band, ...

5. Ideals of a ring of fractions

Let R be a commutative ring and S a multiplicative set. There is an obvious ring homomorphism $\varphi : R \to S^{-1}R$ taking a to $\frac{a}{1}$. (Funny people who don't insist that $1 \in S$ just choose *any* element $s \in S$ and take a to $\frac{sa}{s}$. This works fine, and it doesn't matter which element s is chosen.) We have inclusion-preserving mappings

$$\{\text{ideals of } R\} \xrightarrow{\sigma} \{\text{ideals of } S^{-1}R\}$$
$$\{\text{ideals of } R\} \xleftarrow{\tau} \{\text{ideals of } S^{-1}R\}$$

given by $\sigma(I) = S^{-1}I$ and $\tau(J) = \varphi^{-1}J$, when I is an ideal of R and J is an ideal of $S^{-1}R$.

`ex:ideals` **Exercise 5.1.** Check that $\sigma(\tau(J)) = J$ for every ideal J of $S^{-1}R$. (In particular, σ is surjective and τ is injective. Thus, for example, $S^{-1}R$ is Noetherian if R is Noetherian.) Show that $\tau(\sigma(I)) = \{r \in R \mid sr \in I \text{ for some } s \in S\}$. Show that σ and τ give reciprocal bijections between $\{Q \in \operatorname{Spec} R \mid Q \cap S = \emptyset\}$ and $\operatorname{Spec} S^{-1}R$. (Here $\operatorname{Spec} A$ denotes the set of prime ideals of a commutative ring A.) In particular, if P is a prime ideal of R, then $\operatorname{Spec} R_P$ can be identified with $\{Q \in \operatorname{Spec} R \mid Q \subseteq P\}$. Therefore R_P is a local ring with maximal ideal P_P. But that looks sort of dumb, so we write, instead, PR_P for the maximal ideal of R_P.

`sec:completion`

6. Completion

Let R be a local ring with maximal ideal $\mathfrak{m}$. (Remember: local rings are commutative and Noetherian.) Let M be a finitely generated R-module. The Krull Intersection Theorem (see [**?**, Theorem 8.10]) states that $\bigcap_{n=1}^{\infty} \mathfrak{m}^n M = 0$. Thus, if x and y are distinct elements of M, there is a largest integer n for which $x - y \in \mathfrak{m}^n M$, and we define $d(x, y)$, the *distance* between x and y, to be 2^{-n}. Defining $d(x, x) = 0$, we make M into a metric space. (Exercise: Check this!) Borrowing terminology from basic analysis, we say that M is *complete* provided every Cauchy sequence

sequence in M converges. If R is complete, one can show that every finitely generated R-module M is also complete. (Prove this, filling in the details of the proof of [**?**, Proposition 2.9].)

Now, given an arbitrary local ring R and a finitely generated R-module M, one can form their completions $\widehat{R}$ and $\widehat{M}$ and show easily that $\widehat{R}$ has a natural structure of quasi-local ring with maximal ideal $\widehat{\mathfrak{m}}$, and that $\widehat{M}$ is a finitely generated $\widehat{R}$-module. In fact (and this is far from trivial), it turns out that $\widehat{R}$ is Noetherian, i.e., it's a complete local ring.

For our purposes, the wonderful thing about complete local rings is that their finitely generated modules satisfy KRS (as in my previous talk). That is, every finitely generated module over a complete local ring is uniquely (up to isomorphism and permutation of the summands) a direct sum of indecomposable modules. This was first proved by Swan in the 1960s. The proof is not easy, but an almost self-contained proof can be found in [**?**]. The results discussed in my previous talk (on silly direct-sum decompositions) depend on an analysis of the map $[M] \mapsto [\widehat{M}]$ from R-mod to $\widehat{R}$-mod. It is well-known that this map is injective, that is, two finitely generated R-modules that become isomorphic after completion must already be isomorphic. One needs, however, a much stronger result: If M and N are finitely generated R-modules and $\widehat{M} \mid \widehat{N}$, then $M \mid N$. (Recall: "$M \mid N$" means "M is isomorphic to a direct summand of N".) This is one of the main ingredients in the proof that $\operatorname{add} M$ is Diophantine. The proof of the realization theorem uses these ideas, together with a very technical construction of indecomposable finitely generated modules over the completion. We refer the interested reader to [**?**, (2.3) and (2.4)] for the details. For an exposition of all of the details *except* this technical construction, and for many of the other things I have talked about today, please consult the expository paper [**?**].

References

[1] N. Baeth and R. Wiegand, Factorization theory and decompositions of modules, *Amer. Math. Monthly* **1**20 (2013), 3–34.

[2] E. G. Evans, Jr., Krull-Schmidt and cancellation over local rings, *Pacific J. Math.* **4**6 (1973), 115–121.

[3] H. Matsumura *Commutative Ring Theory*, Cambridge Studies in Advanced Math. **8** (1986), Cambridge University Press.

[4] R. Wiegand Direct-sum decompositions over local rings, *J. Algebra* **2**40 (2001), 83–97.