HILBERT POLYNOMIALS OF MULTIGRADED FILTRATIONS OF IDEALS-II

PARANGAMA SARKAR
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Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ and I be an $\mathfrak{m}$-primary ideal of R.

(1) In 1979, S. Goto and Y. Shimoda [3] proved that $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n$ is Cohen-Macaulay if and only if the associated graded ring of I, $G(I) = \bigoplus_{n\geq 0} \frac{I^n}{I^{n+1}}$ is Cohen-Macaulay and the reduction number $r(I)$ is at most $d-1$.

(2) In 1988, C. Huneke and J. D. Sally [6] proved that if $(R, \mathfrak{m})$ is two-dimensional regular local ring and I is a complete $\mathfrak{m}$-primary ideal, then $\mathcal{R}(I)$ is Cohen-Macaulay.

(3) In 1991, J. K. Verma [14] generalized this result for several ideals and proved that, for $\mathfrak{m}$-primary complete ideals $I_1, \ldots, I_s$ in a two-dimensional regular local ring $(R, \mathfrak{m})$, $\mathcal{R}(I_1, \ldots, I_s)$ is Cohen-Macaulay.

(4) M. Herrmann, E. Hyry, J. Ribbe and Z. Tang [4] proved that if $(R, \mathfrak{m})$ is a Noetherian local ring of dimension two, if $I_1, \ldots, I_s$ are $\mathfrak{m}$-primary ideals of R and for $i \neq j$, the joint reduction number of the filtration $\{\underline{I}^{\underline{n}}\}_{\underline{n}\in\mathbb{Z}^s}$ of type $e_i + e_j$ ($e_i = (0, \ldots, 1, \ldots, 0)$ in $\mathbb{Z}^s$ where 1 occurs at ith position) is zero, and if $\mathcal{R}(I_1, \ldots, I_{s-2})$ is Cohen-Macaulay, then $\mathcal{R}(I_1, \ldots, I_s)$ is Cohen-Macaulay if and only if $\mathcal{R}(I_1, \ldots, I_{s-2}, I_{s-1})$ and $\mathcal{R}(I_1, \ldots, I_{s-2}, I_s)$ are Cohen-Macaulay.

It is well known that, for all $n \gg 0$, the Hilbert function of an $\mathfrak{m}$-primary ideal I, $H_I(n) = \lambda\left(\frac{R}{I^n}\right)$ is the Hilbert polynomial

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I).$$

In 1987, C. Huneke [5] and independently A. Ooishi [13], proved that

$$e_0(I) - e_1(I) = \lambda\left(\frac{R}{I}\right) \text{ if and only if } r(I) \leq 1.$$

In this case $H_{\underline{I}}(n) = P_{\underline{I}}(n)$ for all $n \geq 1$, for $j = 2, \ldots, d$, $e_j(I) = 0$ and $\mathcal{R}(I)$ is Cohen-Macaulay.
We unify all these results and study Cohen-Macaulayness of the Rees algebra of multigraded admissible filtrations .

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1. The Difference Formula

We prove a result analogous to the Grothendieck-Serre formula for $P_{\mathcal{F}}(\underline{n}) - H_{\mathcal{F}}(\underline{n})$ in terms of local cohomology modules of $\mathcal{R}'(\mathcal{F})$. This generalizes results of Johnston-Verma [9], Blancafort [1], Jayanthan-Verma [7].

Proposition 1.1. *Let S' be a $\mathbb{Z}^s$-graded ring and $S = \bigoplus_{\underline{n}\in\mathbb{N}^s} S'_{\underline{n}}$. Then $H^i_{S_{++}}(S') \cong H^i_{S_{++}}(S)$ for all $i > 1$ and the sequence*

$$0 \longrightarrow H^0_{S_{++}}(S) \longrightarrow H^0_{S_{++}}(S') \longrightarrow \frac{S'}{S} \longrightarrow H^1_{S_{++}}(S) \longrightarrow H^1_{S_{++}}(S') \longrightarrow 0$$

is exact.

othendieck-Serre formula

Theorem 1.2. [11, Theorem 4.3] *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then*

(1) $\lambda_R\left(H^i_{\mathcal{R}(\underline{I})_{++}}(\mathcal{R}'(\mathcal{F}))_{\underline{n}}\right) < \infty$ for all $i \geq 0$ and $\underline{n} \in \mathbb{Z}^s$.

(2) $P_{\mathcal{F}}(\underline{n}) - H_{\mathcal{F}}(\underline{n}) = \sum_{i\geq 0}(-1)^i\lambda\left(H^i_{\mathcal{R}(\underline{I})_{++}}(\mathcal{R}'(\mathcal{F}))_{\underline{n}}\right)$ for all $\underline{n} \in \mathbb{Z}^s$.

2. Analogues of the Huneke-Ooishi Theorem

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$, and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then

(1) the filtration $\mathcal{F}^{\Delta} = \{\mathcal{F}(ne)\}_{n\in\mathbb{Z}}$ is $I_1 \cdots I_s$-admissible,
(2) the filtration $\mathcal{F}^{(i)} = \{\mathcal{F}(ne_i)\}_{n\in\mathbb{Z}}$ is I_i-admissible for all $i = 1, \ldots, s$.

For the Hilbert polynomial of $\mathcal{F}^{(i)}$, we write

$$P_{\mathcal{F}^{(i)}}(x) = e_0(\mathcal{F}^{(i)})\binom{x+d-1}{d} - e_1(\mathcal{F}^{(i)})\binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{F}^{(i)}).$$

Huneke-Ooishi

Theorem 2.1. [11, Theorem 5.5] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$, and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then for all $i = 1, \ldots, s$,*
(1) $e_{(d-1)e_i}(\mathcal{F}) \geq e_1(\mathcal{F}^{(i)})$,
(2) $e(I_i) - e_{(d-1)e_i}(\mathcal{F}) \leq \lambda(R/\mathcal{F}(e_i))$,
(3) $e(I_i) - e_{(d-1)e_i}(\mathcal{F}) = \lambda(R/\mathcal{F}(e_i))$ *if and only if* $r(\mathcal{F}^{(i)}) \leq 1$ *and* $e_{(d-1)e_i}(\mathcal{F}) = e_1(\mathcal{F}^{(i)})$.

3. Local cohomology of $\mathcal{R}(\mathcal{F})$

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension two, let I, J be $\mathfrak{m}$-primary ideals, and let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^2}$ be an (I, J)-admissible filtration of ideals in R. Suppose $e_{e_i}(\mathcal{F}) = e_1(\mathcal{F}^{(i)})$ for $i = 1, 2$. Then we proved [11, Theorem 6.6] the following are equivalent:

(1) For every joint reduction (a, b) of $\mathcal{F}$ of type e, $[H^2_{(at_1, bt_2)}(\mathcal{R}'(\mathcal{F}))]_{(0,0)} = 0$,
(2) $e_2(\mathcal{F}^{\Delta}) = e_2(\mathcal{F}^{(1)}) + e_2(\mathcal{F}^{(2)})$,
(3) the joint reduction number of $\mathcal{F}$ of type e is zero.

The a-invariants of $\mathcal{R}(\mathcal{F})$ as $\mathcal{R}(\underline{I})$-modules defined by

$$a^1(\mathcal{R}(\mathcal{F})) = \sup\{k \in \mathbb{Z} \mid [H_{\mathcal{M}}^{\dim \mathcal{R}(\mathcal{F})}(\mathcal{R}(\mathcal{F}))]_{(k,q)} \neq 0 \text{ for some } q \in \mathbb{Z}\}$$
$$a^2(\mathcal{R}(\mathcal{F})) = \sup\{k \in \mathbb{Z} \mid [H_{\mathcal{M}}^{\dim \mathcal{R}(\mathcal{F})}(\mathcal{R}(\mathcal{F}))]_{(p,k)} \neq 0 \text{ for some } p \in \mathbb{Z}\}$$

are -1 where $\mathcal{M}$ is the maximal homogeneous ideal of $\mathcal{R}(\underline{I})$ [12].

4. Cohen-Macaulay Property of the Rees algebra

In this section we relate the Cohen-Macaulayness of the bigraded Rees algebra $\mathcal{R}(\mathcal{F})$ with Hilbert coefficients, reduction numbers and joint reduction numbers. Let I, J be $\mathfrak{m}$-primary ideals in a Cohen-Macaulay local ring $(R, \mathfrak{m})$ of dimension two with infinite residue field, and let $\mathcal{F}$ be a $\mathbb{Z}^2$-graded (I, J)-admissible filtration of ideals in R. M. Herrmann, E. Hyry, J. Ribbe and Z. Tang [4] proved that, if R is a Cohen-Macaulay local ring and the joint reduction number of $\{I^r J^s\}_{r,s\in\mathbb{Z}}$ of type e is zero, then $\mathcal{R}(I, J)$ is Cohen-Macaulay if and only if $\mathcal{R}(I)$ and $\mathcal{R}(J)$ are Cohen-Macaulay. We showed that this is also valid for bigraded filtrations.

Theorem 4.1. [11, Theorem 7.1] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension two, I, J be $\mathfrak{m}$-primary ideals and $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^2}$ be an (I, J)-admissible filtration of ideals in R. If the joint reduction number of $\mathcal{F}$ with respect to some joint reduction of type e is zero, then $\mathcal{R}(\mathcal{F})$ is Cohen-Macaulay if and only if $\mathcal{R}(\mathcal{F}^{(i)}) = \bigoplus_{n\geq 0} \mathcal{F}(ne_i)t^n$ is Cohen-Macaulay for all $i = 1, 2$.*

Main Theorem

Theorem 4.2. [11, Theorem 7.3] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension two, let I_1, I_2 be $\mathfrak{m}$-primary ideals of R, and let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^2}$ be an (I_1, I_2)-admissible filtration of ideals in R. Then the following statements are equivalent.*

(1) $e(I_1) - e_{e_1}(\mathcal{F}) = \lambda(R/\mathcal{F}^{(1)})$ and $e(I_2) - e_{e_2}(\mathcal{F}) = \lambda(R/\mathcal{F}^{(2)})$,
(2) $r(\mathcal{F}^{(1)}), r(\mathcal{F}^{(2)}) \leq 1$ and the joint reduction number of $\mathcal{F}$ of type e is zero,
(3) $P_{\mathcal{F}}(r, s) = H_{\mathcal{F}}(r, s)$ for all $r, s \geq 0$,
(4) $\mathcal{R}(\mathcal{F})$ is Cohen-Macaulay.

Proof. (1) $\Rightarrow$ (2) : Fix $i \in \{1, 2\}$. Since $e(I_i) - e_{e_i}(\mathcal{F}) = \lambda(R/\mathcal{F}^{(i)})$, we have $e_{e_i}(\mathcal{F}) = e_1(\mathcal{F}^{(i)})$ and $r(\mathcal{F}^{(i)}) \leq 1$. Then by [1, Theorem 4.3.6], $G(\mathcal{F}^{(i)})$ is Cohen-Macaulay and $e_2(\mathcal{F}^{(i)}) = 0$. Using [10, Proposition 3.23], we get $e_2(\mathcal{F}^{\Delta}) = 0$. Thus joint reduction number of $\mathcal{F}$ of type e is zero.
(2) $\Rightarrow$ (3) : One can show that the joint reduction number of $\mathcal{F}$ of type e is zero if and only if $\lambda(R/\mathcal{F}(r, s)) = rse(I|J) + \lambda(R/\mathcal{F}(r, 0)) + \lambda(R/\mathcal{F}(0, s))$ for all $r, s \geq 1$. If $r = 0$ or $s = 0$ the above equation is still true.
Fix $i \in \{1, 2\}$. Then $r(\mathcal{F}^{(i)}) \leq 1$ implies $P_{\mathcal{F}^{(i)}}(n) = \lambda(R/\mathcal{F}(ne_i))$ for all $n \geq 0$. Therefore $\lambda(R/\mathcal{F}(r, s)) = rse(I_1|I_2) + P_{\mathcal{F}^{(1)}}(r) + P_{\mathcal{F}^{(2)}}(s)$ for all $r, s \geq 0$.
(3) $\Rightarrow$ (1) $\Rightarrow$ (4) : Since $P_{\mathcal{F}}(r, s) = \lambda(R/\mathcal{F}(r, s))$ for all $r, s \geq 0$, the conditions of (1) are satisfied. Hence $r(\mathcal{F}^{(1)}), r(\mathcal{F}^{(2)}) \leq 1$, and the joint reduction number of $\mathcal{F}$ of type e is zero. Thus $\mathcal{R}(\mathcal{F}^{(1)})$ and $\mathcal{R}(\mathcal{F}^{(2)})$ are Cohen-Macaulay, by [15, Theorem 2.3]. Hence $\mathcal{R}(\mathcal{F})$ is Cohen-Macaulay.
(4) $\Rightarrow$ (1) : Since $\mathcal{R}(\mathcal{F})$ is Cohen-Macaulay and $a^j(\mathcal{R}(\mathcal{F})) = -1$ for all $j = 1, 2$, we have $[H^i_{\mathcal{R}(\underline{I})_{++}}(\mathcal{R}(\mathcal{F}))]_{(r,s)} = 0$ for all $r, s \geq 0$ and $i = 0, 1, 2$. Therefore $[H^i_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{(r,s)} \simeq [H^i_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))]_{(r,s)} = 0$ for all $r, s \geq 0$ and $i = 0, 1, 2$. Therefore, by the Difference formula (1.2), we have $P_{\mathcal{F}}(r, s) = \lambda(R/\mathcal{F}(r, s))$ for

all $r,s \geq 0$. Taking $r = s = 0$, $(r,s) = (1,0)$ and $(r,s) = (0,1)$ in the previous equality, we get the desired result. $\square$

The following examples are in [7].

Example 4.3. Let $A = \mathbb{C}[[x,y]]$. Consider the plane curve $f = y^2 - x^n$ and $\mathfrak{m} = (x,y)A$. Let J be the Jacobian ideal of $f = 0$ and $\mathcal{F} = \{\mathfrak{m}^r J^s\}_{r,s\in\mathbb{Z}}$. Then $r(J) = r(\mathfrak{m}) = 0$, $y\mathfrak{m} + xJ = \mathfrak{m}J$ and $P_{\mathcal{F}}(r,s) = \binom{r+1}{2} + rs + (n-1)\binom{s+1}{2}$.

Example 4.4. Neither of the conditions in (1) of Theorem 4.2 can be dropped to get the conclusions (2) and (3). Let $(R,\mathfrak{m})$ be a regular local ring of dimension 2. Let $\mathcal{F} = \{\mathfrak{m}^r I^s\}_{r,s\in\mathbb{Z}}$ where $\mathfrak{m} = (x,y)$ and $I = (x^3, x^2y^4, xy^5, y^7)$. Then $I\mathfrak{m} = x^3\mathfrak{m} + yI$. Using Macaulay 2 [2], $\lambda\left(\frac{R}{I}\right) = 16$. The ideal $J = (x^3, y^7)$ is a minimal reduction of I and $JI^2 = I^3$ but $JI \neq I^2$. The Hilbert polynomial of $\mathcal{F}$ is $P_{\mathcal{F}}(r,s) = \binom{r+1}{2} + 3rs + 21\binom{s+1}{2} - 6s + 1$ which shows $e(I) - e_{(0,1)}(\mathcal{F}) = 15 < \lambda\left(\frac{R}{I}\right)$.

Example 4.5. Theorem 4.2 is not true for dimension greater than 2. Let $A = k[[x,y,z]]$, $I = \mathfrak{m}^2 + (z)$, $J = (x, y^3, z)$ and $\mathcal{F} = \{I^r J^s\}_{r,s\in\mathbb{Z}}$. Then $r(J) = 0$, (x^2, y^2, z) is a reduction of I with $r(I) = 1$ and $IJ = (x,z)I + y^2J = xI + (y^2,z)J$. By computations on Macaulay 2 [2], we get $\operatorname{depth}(\mathcal{R}(\mathcal{F})) < \dim(\mathcal{R}(\mathcal{F})) = 5$. Therefore $\mathcal{R}(\mathcal{F})$ is not Cohen-Macaulay.

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