HILBERT POLYNOMIALS OF MULTIGRADED FILTRATIONS OF IDEALS-I

PARANGAMA SARKAR
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Let $(R, \mathfrak{m})$ be a Noetherian local ring and I be an ideal of R. A **Reduction** of an ideal I is an ideal $J \subseteq I$ such that $JI^n = I^{n+1}$ for some n. We say J is a **minimal reduction** of I if whenever $K \subseteq J$ and K is a reduction of I, then $K = J$. We define the **reduction number of I with respect to J** is

$$r_J(I) = \min\{m : JI^n = I^{n+1} \text{ for } n \geq m\}$$

and **reduction number of I** is

$$r(I) = \min\{r_J(I) : J \text{ is a minimal reduction of } I\}.$$

Let I be an $\mathfrak{m}$-primary ideal of R. In [**?**], P. Samuel showed that for $n \gg 0$, the Hilbert function $H_I(n) = \lambda\left(\frac{R}{I^n}\right)$ coincides with a degree d polynomial

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I),$$

called the Hilbert polynomial of I. The coefficients $e_i(I)$ are integers, $e_0(I)$ is called the multiplicity of I and denoted by $e(I)$. For any minimal reduction J of an $\mathfrak{m}$-primary ideal I, $e(I) = e(J)$.

1. Multigraded filtrations

Let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R.

Definition 1.1. A set of ideals $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ is called a $\mathbb{Z}^s$-graded **$\underline{I}$-filtration** if for all $\underline{m}, \underline{n} \in \mathbb{Z}^s$, (i) $\mathcal{F}(\underline{n})\mathcal{F}(\underline{m}) \subseteq \mathcal{F}(\underline{n} + \underline{m})$, (ii) $\underline{I}^{\underline{n}} \subseteq \mathcal{F}(\underline{n})$ and if $\underline{m} \geq \underline{n}$, (iii) $\mathcal{F}(\underline{m}) \subseteq \mathcal{F}(\underline{n})$.

Let $t_1, \ldots, t_s$ be indeterminates. For $\underline{n} \in \mathbb{Z}^s$, put $\underline{t}^{\underline{n}} = t_1^{n_1} \cdots t_s^{n_s}$ and denote the Rees ring of $\mathcal{F}$ by $\mathcal{R}(\mathcal{F}) = \bigoplus_{\underline{n} \in \mathbb{N}^s} \mathcal{F}(\underline{n})\underline{t}^{\underline{n}}$ and the extended Rees ring of $\mathcal{F}$ by $\mathcal{R}'(\mathcal{F}) = \bigoplus_{\underline{n} \in \mathbb{Z}^s} \mathcal{F}(\underline{n})\underline{t}^{\underline{n}}$. For $\underline{n} \in \mathbb{Z}^s$, we denote $\underline{n}^+ = (n_1^+, \ldots, n_s^+)$ where $n_i^+ = \max\{0, n_i\}$.

Definition 1.2. A $\mathbb{Z}^s$-graded filtration $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ of ideals in R is called an $\underline{I} = (I_1, \ldots, I_s)$**-admissible filtration** if $\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n}^+)$ for all $\underline{n} \in \mathbb{Z}^s$ and $\mathcal{R}'(\mathcal{F})$ is a finite $\mathcal{R}'(\underline{I})$-module.

Example 1.3. The filtration $\{I_1^{n_1} \cdots I_s^{n_s}\}_{n_i \in \mathbb{Z}}$ is an $\underline{I}$-admissible filtration.

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2. The Multigraded Hilbert function and polynomial

For an $\underline{I}$-admissible filtration $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ of ideals in a local ring $(R, \mathfrak{m})$ of dimension d, D. Rees showed existence of a polynomial

$$P_{\mathcal{F}}(\underline{n}) = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\in\mathbb{N}^s \\ |\alpha|\leq d}} (-1)^{d-|\alpha|} e_\alpha(\mathcal{F}) \binom{n_1+\alpha_1-1}{\alpha_1} \cdots \binom{n_s+\alpha_s-1}{\alpha_s}$$

of degree d which coincides with the **Hilbert function** $H_{\mathcal{F}}(\underline{n}) = \lambda\left(\frac{R}{\mathcal{F}(\underline{n})}\right)$ for all large $\underline{n}$ [?]. The coefficients $e_\alpha(\mathcal{F})$ are integers and the coefficients $e_\alpha(\mathcal{F})$ where $\alpha_1 + \cdots + \alpha_s = d$ are called mixed multiplicities of $\mathcal{F}$. B. Teissier [?] proved the existence of the polynomial for the filtration $\{I_1^{n_1} \cdots I_s^{n_s}\}_{n_1,\ldots,n_s\in\mathbb{Z}}$. This was proved by P. B. Bhattacharya for the filtration $\{I_1^{n_1} I_2^{n_2}\}_{n_1,n_2\in\mathbb{Z}}$ and $s = 2$ in [?].

3. Joint reduction of multigraded filtrations

Rees [?] introduced the concept of joint reduction for the filtration $\{I_1^{n_1} \cdots I_s^{n_s}\}_{n_1,\ldots,n_s\in\mathbb{Z}}$. A set of elements $\{x_1, \ldots, x_d\}$ is a joint reduction of $\{I_1, \ldots, I_d\}$ if $x_i \in I_i$ for each i and $\sum_{i=1}^{d} x_i I_1^{n_1} \cdots I_i^{n_i-1} \cdots I_d^{n_d} = I_1^{n_1} \cdots I_d^{n_d}$ for some $(n_1 \ldots, n_d) \in \mathbb{N}^d$.
Note that if $I = I_1 = \cdots = I_d$ then a joint reduction is just a reduction of I.

Example 3.1. Let $R = k[|x, y|]$, $I = (x, y^2)$ and $J = (x^2, y)$. Then $\{x, y\}$ is a joint reduction of (I, J) and $yI + xJ = IJ$.

Definition 3.2. We define the **joint reduction** of $\mathcal{F}$ of type $\mathbf{q} = (q_1, \ldots, q_s) \in \mathbb{N}^s$ to be a collection of q_i elements $x_{i1}, \ldots, x_{iq_i} \in I_i$ for all $i = 1, \ldots, s$, such that $q_1 + \cdots + q_s = d$ and $\sum_{i=1}^{s} \sum_{j=1}^{q_i} x_{ij}\mathcal{F}(\underline{n} - e_i) = \mathcal{F}(\underline{n})$ for all large $\underline{n} \in \mathbb{N}^s$.

Definition 3.3. We say the **joint reduction number** of $\mathcal{F}$ with respect to a joint reduction $\{x_{ij} \in I_i : j = 1, \ldots, q_i; i = 1, \ldots, s\}$ of type $\mathbf{q}$ is zero if

$$\sum_{i=1}^{s} \sum_{j=1}^{q_i} x_{ij}\mathcal{F}(\underline{n} - e_i) = \mathcal{F}(\underline{n}) \text{ for all } \underline{n} \geq \sum_{i\in A} e_i, \text{ where } A = \{i | q_i \neq 0\},$$

and the joint reduction number of $\mathcal{F}$ of type $\mathbf{q}$ is zero if the joint reduction number of $\mathcal{F}$ with respect to every joint reduction of type $\mathbf{q}$ is zero.

4. Local cohomology of modules

Let R be a Noetherian commutative ring and $\mathfrak{a}$ be an ideal in R. For each R-module M, define $\Gamma_{\mathfrak{a}}(M) = \{m \in M | \mathfrak{a}^t m = 0 \text{ for some } t \in \mathbb{N}\}$. For a homomorphism $f : M \to N$ of R-modules, $f(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$.

(1) $\Gamma_{\mathfrak{a}}(-)$ is a functor on the category of R-modules and it extends to a functor on the the category of complexes of R-modules.
(2) $\Gamma_{\mathfrak{a}}(-)$ is a left exact functor.

Consider an injective resolution of

$$I^\bullet : 0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^i \longrightarrow I^{i+1} \longrightarrow \cdots$$

of M. Apply the functor $\Gamma_{\mathfrak{a}}(-)$ to the complex $I^\bullet$ to obtain

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(I^0) \longrightarrow \cdots \longrightarrow \Gamma_{\mathfrak{a}}(I^i) \longrightarrow \Gamma_{\mathfrak{a}}(I^{i+1}) \longrightarrow \cdots$$

and take the jth cohomology module of this complex. This cohomology module is called the j-th local cohomology of M with support in $\mathfrak{a}$ and denoted by $H^j_{\mathfrak{a}}(M)$.

Example 4.1. Let R be a ring and $\mathfrak{a}$ is an ideal in R such that $\mathfrak{a}^n = 0$ for some $n \geq 0$. Then $\Gamma_{\mathfrak{a}}(-)$ is the identity functor and hence

$$H^j_{\mathfrak{a}}(M) = \begin{cases} M & \text{if } j = 0 \\ 0 & \text{if } j \geq 0. \end{cases}$$

Proposition 4.2. *(1) $H^j_{\mathfrak{a}}(M) = 0$ for all $i > \dim M$.*

(2) Define $ara(\mathfrak{a}) = \min\{k : \exists\ a_1, \ldots, a_k \in R$ such that rad $(a_1, \ldots, a_k) =$ rad $\mathfrak{a}\}$. The number $ara(\mathfrak{a})$ is called the ***arithmetic rank*** *of I. Then $H^j_{\mathfrak{a}}(M) = 0$ for all $i > ara(\mathfrak{a})$.*

(3) If rad $\mathfrak{a} =$ rad $\mathfrak{b}$, then, for each j, $H^j_{\mathfrak{a}}(M) \cong H^j_{\mathfrak{b}}(M)$.

(4) If S is a multiplicative closed subset of R, then $S^{-1}H^j_{\mathfrak{a}}(M) \cong H^j_{\mathfrak{a}}(S^{-1}M)$.

(5) If $R \to S$ is a ring homomorphism and N is an S-module, then

$$H^j_{\mathfrak{a}}(N) \cong H^j_{\mathfrak{a}S}(N).$$

(6) If $R \to S$ is flat, then there is a natural isomorphism of S-modules

$$S \otimes H^j_{\mathfrak{a}}(M) \cong H^j_{\mathfrak{a}S}(S \otimes_R M).$$

Theorem 4.3. *For each R module M and $j \geq 0$, $\varinjlim_t Ext^j_R(R/\mathfrak{a}^t, M) \cong H^j_{\mathfrak{a}}(M)$.*

Let $\mathbf{x} = x_1, \ldots, x_k$ be elements in R, $\mathfrak{a} = (x_1, \ldots, x_k)$ and $\mathbf{x^t} = x_1^t, \ldots, x_k^t$. For each t, let $K(\mathbf{x^t}, M)_\bullet$ denote $M \otimes K(\mathbf{x^t})_\bullet$, the Koszul complex of M with respect to $\mathbf{x^t}$.

Theorem 4.4. *For each R module M and each integer j,*

$$H^j_{\mathfrak{a}}(M) \cong \varinjlim_t H^j(K(\mathbf{x^t}, M)_\bullet).$$

Let $R = \oplus_{\underline{n} \in \mathbb{N}^s} R_{\underline{n}}$ be a graded Noetherian ring and $M = \oplus_{\underline{n} \in \mathbb{Z}^s} M_{\underline{n}}$ be a graded module in the category of graded modules of R, denoted by ${}^*\mathcal{C}(R)$. Let I be a homogeneous ideal of R. Then:

(1) ${}^*\mathcal{C}(R)$ has enough injectives and ${}^*H^j_I(M)$ is called the j-th graded local cohomology module of M with support in I.
(2) If we forget the grading on ${}^*H^j_I(M)$, then the resulting R-module is isomorphic to $H^j_I(M)$.

Theorem 4.5. *For any joint reduction (a, b) of $\mathcal{F}$,*

$$[H^2_{(at_1, bt_2)}(\mathcal{R}'(\mathcal{F}))]_{(0,0)} \cong \varinjlim_k \frac{\mathcal{F}(k,k)}{a^k\mathcal{F}(0,k) + b^k\mathcal{F}(k,0)}.$$

Proof. For any joint reduction (a, b) of $\mathcal{F}$, consider the Koszul complex

$$F^{k\cdot} : 0 \longrightarrow \mathcal{R}'(\mathcal{F}) \xrightarrow{\alpha_k} \mathcal{R}'(\mathcal{F})(ke_1) \oplus \mathcal{R}'(\mathcal{F})(ke_2) \xrightarrow{\beta_k} \mathcal{R}'(\mathcal{F})(ke) \longrightarrow 0,$$

where the maps are defined as,

$$\alpha_k(1) = ((at_1)^k, (bt_2)^k) \text{ and } \beta_k(u, v) = -(bt_2)^k u + (at_1)^k v.$$

Therefore

$$[H^2_{(at_1, bt_2)}(\mathcal{R}'(\mathcal{F}))]_{(0,0)} \cong \varinjlim_k \frac{\mathcal{F}(k,k)}{(\operatorname{im}\beta_k)_{(0,0)}}.$$

Since $(\operatorname{im}\beta_k)_{(0,0)} = a^k\mathcal{F}(0,k) + b^k\mathcal{F}(k,0)$, we get the required result. □

5. A few definitions

(1) A sequence $a_1, a_2, \ldots, a_n$ of elements in a proper ideal I of R is called a **regular sequence** if a_i is a nonzerodivisor on $R/(a_1, a_2, \ldots, a_{i-1})$ for $i = 1, 2, \ldots, n$.
(2) All maximal regular sequences in I have equal length called the **grade of** I and it is denoted by $grade(I)$.
(3) The grade of $\mathfrak{m}$ is called the **depth of** R.
(4) For any local ring R, $\operatorname{depth} R \leq \dim R$. We say that R is **Cohen-Macaulay** if $\operatorname{depth} R = \dim R$.
(5) A Noetherian local ring $(R, \mathfrak{m})$ is called **regular** if $\dim R$ is same as number of minimal generators of $\mathfrak{m}$.
(6) A Noetherian local ring is called analytically unramified if its completion has no nonzero nilpotent element.

6. Rees' Theorem and its consequences

Theorem 6.1 ([?]). *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension 2. Let I, J be $\mathfrak{m}$-primary ideals and let (a, b) be a good joint reduction of the filtration $\{\overline{I^rJ^s}\}_{r,s\in\mathbb{Z}}$. Then following are equivalent.*

(1) $\bar{e}_2(IJ) = \bar{e}_2(I) + \bar{e}_2(J)$.
(2) for all $r, s > 0$, $\overline{I^rJ^s} = a\overline{I^{r-1}J^s} + b\overline{I^rJ^{s-1}}$.

Theorem 6.2. *Under the assumptions above,* $\overline{IJ} = \overline{I}\ \overline{J}$ *if* $\overline{IJ} = a\overline{J} + b\overline{I}$.

Proof. $\overline{IJ} = a\overline{J} + b\overline{I}$ implies that $\overline{IJ} \subseteq \overline{I}\ \overline{J}$. Since $\overline{I}\ \overline{J} \subseteq \overline{IJ}$ in a Noetherian ring, $\overline{IJ} = \overline{I}\ \overline{J}$. □

Theorem 6.3 ([?]). *Let $(R, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay analytically unramified local ring with infinite residue field and let I be an $\mathfrak{m}$-primary ideal. Then the following are equivalent.*

(1) $\bar{e}_2(I) = 0$.
(2) $\overline{I^n} = (x, y)\overline{I^{n-1}}$ *for* $n \geq 2$ *and for any minimal reduction* (x, y) *of* I.

Theorem 6.4 (Zariski). *Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring and let I, J be integrally closed ideals in R. Then IJ is integrally closed.*

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