# BOREL-FIXED IDEALS AND THEIR FREE RESOLUTIONS

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ABSTRACT. This is an extended abstract of the talk at the First International Workshop and Conference in Commutative Algebra, held at Tribhuvan University, Kathmandu, Nepal during April 20 - 26, 2015. It is written in an expository style. We first look at Borel-fixed ideals and a minimal free resolution in characteristic zero, due to Eliahou and Kervaire. We will then consider a related conjecture of Pardue in characteristic p, and discuss a counter-example to this conjecture, obtained in joint work with G. Caviglia.

## 1. INTRODUCTION

Let $\Bbbk$ be a field and $R := \Bbbk[x_1, \dots, x_n]$ a polynomial ring where $x_1, \dots x_n$ are indeterminates over $\Bbbk$. The *general linear group* $\mathrm{GL}_n(\Bbbk)$ (which we think of as invertible $n \times n$ matrices over $\Bbbk$) acts on $R$ as follows: First, if

$$g = [a_{i,j}]_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \in \mathrm{GL}_n(\Bbbk)$$

then let

$$gx_i := \sum_{j=1}^{n} a_{i,j} x_j.$$

For a monomial $m := x_1^{e_1} \cdots x_n^{e_n}$ of $R$, we let

$$gm := (gx_1)^{e_1} \cdots (gx_n)^{e_n}$$

and for a polynomial $f = \alpha_1 m_1 + \cdots + \alpha_r m_r$ (where the $\alpha_i$ are from $\Bbbk$ and the $m_i$ are monomials), we set

$$gf := \sum_{i=1}^{r} \alpha_i (gm_i).$$

This defines a $\Bbbk$-algebra automorphism of $R$.

Let $B$ be the subgroup of $\mathrm{GL}_n(\Bbbk)$ consisting of all the lower-triangular (invertible) matrices. This is a Borel-subgroup of $\mathrm{GL}_n(\Bbbk)$. Note that for every $g = [a_{i,j}]_{1 \leq j \leq i \leq n} \in B$, $gx_i = \sum_{j=1}^{i} a_{i,j} x_j$ is a homogeneous linear polynomial involving $x_1, \dots, x_i$. Since $g$ is invertible and lower-triangular, the diagonal entries must be non-zero.

**Definition 1.1.** An $R$-ideal $I$ is said to be *Borel-fixed* if $gf \in I$ for every $g \in B$ and $f \in I$.

It is easy to observe that a Borel-fixed ideal must be a monomial ideal. More precisely, an $R$-ideal $I$ is monomial if and only if $gf \in I$ for every diagonal matrix $g$ and every $f \in I$. Therefore being Borel-fixed imposes certain conditions on $I$.

**Theorem 1.2** (See [**?**, Theorem 15.23]). *Let $p$ be the characteristic of $\Bbbk$. An $R$-ideal $I$ is Borel-fixed if and only if $J$ is a monomial ideal and the following holds for every monomial minimal generator of $I$: If $x_j^t$ divides $m$ and $x_j^{t+1}$ does not, then for every $i < j$ and $s \prec_p t$, $(x_i/x_j)^s m \in I$.*

Here the partial order $\prec_p$ on positive integers is defined as follows: If $p = 0$, then $s \prec_p t$ if $s \leq t$. If $p > 0$ is a prime number, then for a positive integer $s$, write $s = \sum_{i=0}^{\infty} s_i p^i$ with the $s_i \geq 0$ (uniquely determined). Then $s \prec_p t$ if $s_i \leq t_i$ for all $i \geq 0$.

We now discuss free resolutions of graded $R$-modules. Note that $R$ is a graded ring with $\deg x_i = 1$ for all $i$. Write $\mathfrak{m} = (x_1, \ldots, x_n)$. Let $M$ be a finitely generated graded $R$-module. Then by the Hilbert Syzygy Theorem, there exists a complex

$$0 \longrightarrow F_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0$$

of finitely generated graded free $R$-modules whose homology is concentrated at position 0 and is isomorphic to $M$ (i.e. $\operatorname{coker} \partial_1 \simeq M$). Such a complex is called a *graded free resolution* of $M$. If $\operatorname{Im} \partial_i \subseteq \mathfrak{m} F_{i-1}$ for all $i$, then we say that the resolution is *minimal*. Hereafter, by ideal, module etc, we mean graded ideal, graded module etc.

The *($\mathbb{N}$-graded) Betti numbers* of $M$ are the integers $\beta_{i,j}(M) := \dim_{\Bbbk} \operatorname{Tor}_i^R(M, \Bbbk)_j$. Since one can compute these Tor modules using any free resolution of $M$, one sees immediately that $\beta_{i,j}(M)$ is the number of elements of degree $j$ in any $R$-basis of $F_i$ consisting of homogeneous elements, for any minimal resolution $F_\bullet$. The *Betti table* $\beta(M)$ of $M$ is the collection $\left(\beta_{i,j}(M)\right)_{i,j}$.

There is a natural construction called the *mapping cone* that inductively constructs a free resolution of an ideal (more precisely of the corresponding quotient ring). Let $J$ be an $R$-ideal and $f \in R$ a homogeneous element of degree $d$, suppose that we can construct resolutions $F_\bullet$ of $R/J$ and $G_\bullet$ of $R/(J : f)$. Then the injective map (of degree zero)

$$\frac{R}{(J : f)}(-d) \longrightarrow \frac{R}{J}$$

gives rise to an map of complexes

$$\phi : G_\bullet \otimes_R R(-d) \longrightarrow F_\bullet.$$

The *mapping cone* of $\phi$ is a complex $M(\phi)_\bullet$ with

$$M(\phi)_i = F_i \oplus G_{i-1}$$

and with differentials suitably defined. It is a graded free resolution of $R/(J + (f))$. It is not a minimal free resolution, in general. See [**?**, Appendix A3] for details.

Now suppose that $\Bbbk$ is an infinite field and that its characteristic is zero. If $I$ is a Borel-fixed ideal in $R$, then there is a minimal free resolution of $I$ due to Eliahou and Kervaire. It is inductively constructed using mapping cones, but the key observation is that the minimal monomial generating set of $I$ has an ordering $m_1, \ldots, m_r$ such that each ideal $J_i := (m_1, \ldots, m_i)$ is Borel-fixed, $(J_i : m_{i+1})$ is of the form $(x_1, \ldots, x_{t_i})$ for some $t_i$ and that the mapping cone is minimal. Thus we get a minimal graded free resolution of $I$.

## 2. Pardue's conjecture

**Definition 2.1.** Let $p$ be zero or a prime number. A monomial ideal $J$ of $S := \mathbb{Z}[x_1, \ldots, x_n]$ is said to be *$p$-Borel-fixed* if $J(S \otimes_{\mathbb{Z}} \Bbbk)$ is a Borel-fixed ideal of $(S \otimes_{\mathbb{Z}} \Bbbk)$ for every infinite field $\Bbbk$ of characteristic $p$. □

If $J$ is 0-Borel-fixed, then its Eliahou-Kervaire resolution, constructed using mapping cones, is a minimal resolution of $(S/J) \otimes_{\mathbb{Z}} \ell$ for every field $\ell$, and hence $\beta((S/J) \otimes_{\mathbb{Z}} \ell)$ does not depend on $\ell$. K. Pardue [**?**, p.43] conjectured that this holds also for $p$-Borel-fixed ideals for prime numbers $p$.

**Theorem 2.2** ([**?**, Theorem 3.2])**.** *Let $p$ be any prime. Then there exists a $p$-Borel-fixed $S$-ideal $I$ such that for any field $\ell$, there is a region (independent of $\ell$) of $\beta((S/I) \otimes_{\mathbb{Z}} \ell)$ that is determined by $\beta((S/J) \otimes_{\mathbb{Z}} \ell)$.*

See [**?**, Theorem 3.2] for the precise statement. This shows that, homologically, the class of Borel-fixed ideals in positive characteristic is as bad as the class of all monomial ideals. In particular, beginning with an $S$-ideal $J$ such that $\beta((S/J) \otimes_{\mathbb{Z}} \ell)$ depends on $\ell$ (equivalently, the characteristic of $\ell$), we conclude that there exists a $p$-Borel-fixed ideal $I$ such that $\beta((S/J) \otimes_{\mathbb{Z}} \ell)$

depends on $\ell$, settling Pardue's conjecture in the negative. See [**?**, Example 3.5] for an example using the real projective plane.

## References

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