ASSOCIATED PRIMES AND COTILTING APPROXIMATIONS OVER COMMUTATIVE NOETHERIAN RINGS

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1. Cotilting modules and classes

Cotilting modules were first studied in the representation theory of algebras. They can be viewed as generalizations of injective cogenerators:

Definition 1.1. Let R be a ring and $n < \omega$. A module C is *n-cotilting* provided that

(C1) C has injective dimension at most n.
(C2) $\mathrm{Ext}^i_R(C^X, C) = 0$ for each set X and each $1 \leq i \leq n$.
(C3) There are an injective cogenerator W and a long exact sequence $0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to W \to 0$, with $C_i \in \mathrm{Prod}(C)$, where $\mathrm{Prod}(C)$ denotes the class of all direct summands of (possibly infinite) direct products of copies of C.

The class $\mathcal{C}_C = {}^{\perp}\{C\} = \{M \in \text{Mod–}R \mid \mathrm{Ext}^i_R(M, C) = 0 \text{ for all } i \geq 1\}$ is the *n-cotilting class* induced by C.

Two cotilting modules C and C' are said to be *equivalent*, if ${}^{\perp}\{C\} = {}^{\perp}\{C'\}$.

C is a *minimal* n-cotilting module in case C is a direct summand of each cotilting module equivalent to C.

It is easy to see that the 0-cotilting modules are exactly the injective cogenerators, so there is only one 0-cotilting class of R-modules, namely the class of all R-modules. Also, if R is commutative and noetherian, then the minimal injective cogenerator $W_{min} = \bigoplus_{m \in \mathrm{mSpec}(R)} E(R/m)$ is a minimal 0-cotilting module in the sense of Definition 1.1.

We start with a key property of general cotilting modules:

Theorem 1.2 (Šťovíček)**.** *Every cotilting module C is* pure-injective*, i.e., the functor $Hom_R(-, C)$ is exact on all pure-exact sequences in* Mod–R*.*

In particular, if a minimal n-cotilting module inducing an n-cotilting class $\mathcal{C}$ exists, then, being pure-injective, it is unique up to isomorphism (by a classic result of Bumby).

Since pure-injective modules over several important classses of rings are known, Theorem 1.2 makes it possible to classify all cotilting modules over those rings. We demostrate this on the example of Dedekind (= hereditary) domains:

Example 1.3. Indecomposable pure-injective modules over a Dedekind domain R with quotient field Q are well known. These are, for each $p \in \mathrm{mSpec}(R)$, the

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modules R_{p^n} (= the factors of R modulo p^n for $0 < n < \omega$), the Prüfer modules R_{p^∞} (= the directed unions of R_{p^n} for $n < \omega$), the p-adic modules J_p (= the modules isomorphic to $\mathrm{End}(R_{p^\infty})$), and also Q viewed as an R-module.

Using this classification, one can show that cotilting (= 1-cotilting) modules are parametrized up to equivalence by subsets of $\mathrm{mSpec}(R)$. Given such a subset S, a cotilting module C_S can be defined by the formula

$$C_S = Q \oplus \bigoplus_{p \in S} J_p \oplus \prod_{q \in \mathrm{mSpec}(R) \setminus S} R_{q^\infty}.$$

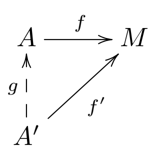
The class $\mathcal{C}_S$ induced by C_S is the class of all S-torsion-free modules, i.e.,

$$\mathcal{C}_S = \{M \in \text{Mod–}R \mid m.p = 0 \text{ implies } m = 0 \text{ for all } m \in M \text{ and } p \in S\}.$$

2. Approximations of modules

Next we present the relation between cotilting and approximations of modules. The following basic notions are due to Enochs and Auslander [2]:

Definition 2.1. (1) A class of modules $\mathcal{A}$ is *covering* in case $\mathcal{A}$ is *precovering*, that is, for each module M there exists $f \in \mathrm{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \mathrm{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ has a factorization through f,



and moreover, f is a $\mathcal{A}$-*cover* of M, that is, for $f = f'$, each such factorization g is an automorphism.

(2) Dually, we define *envelopes* and *enveloping* classses of modules.

Note: Covers and envelopes are unique up to isomorphism.

Example 2.2. The class $\mathcal{I}_0$ of all injective modules is enveloping, since the inclusion $M \hookrightarrow E(M)$ is an $\mathcal{I}_0$-envelope for each module M.

(Pre-)covering and (pre-)enveloping classes make it possible to develop relative homological algebra in the spirit of [2], by using these classes in place of the classes of projective and injective modules, respectively. The following result is relevant to this context:

Proposition 2.3. *Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C}$ is 1-cotilting if and only if $\mathcal{C}$ is a torsion-free class which is covering.*

In general, n-cotilting classes are characterized as follows:

Theorem 2.4. *Let $\mathcal{C}$ be a class of modules and $n < \omega$. Then $\mathcal{C}$ is n-cotilting if and only if*

(1) $\mathcal{C} = {}^\perp\mathcal{E}$ for a class of (pure-injective) modules $\mathcal{E}$,

(2) $\mathcal{C}$ is closed under direct products (and it is covering), and

(3) the class $\mathcal{C}^\perp = \{M \in \text{Mod–}R \mid \mathrm{Ext}^i_R(C, M) = 0$ for all $i \geq 1$ and all $C \in \mathcal{C}\}$ consists of modules of injective dimension $\leq n$.

3. Structure of cotilting classes and modules

From now on, we will assume that R is a commutative noetherian ring.

We start with a classification of 1-cotilting classes in this setting:

Theorem 3.1. *(1) Let P be a lower subset of* $\operatorname{Spec}(R)$ *containing* $\operatorname{Ass}(R)$*, and* $\mathcal{C}_P = \{M \in \text{Mod–}R \mid \operatorname{Ass}(M) \subseteq P\}$*. Then $\mathcal{C}_P$ is a 1-cotilting class.*

(2) Each 1-cotilting class arises as in (1). That is, 1-cotilting classes are parametrized by the lower subsets of $\operatorname{Spec}(R)$ *containing* $\operatorname{Ass}(R)$*.*

Proof. We sketch only the proof of part (1). Let $Q = \operatorname{Spec}(R) \setminus P$. By the results on morphisms between indecomposable injective modules presented in the Preparatory Talk, $\mathcal{C}_P = \{M \in \text{Mod–}R \mid \operatorname{Hom}_R(M, E(R/q)) = 0 \text{ for each } q \in Q\}$. In particular, $\mathcal{C}_P$ is a torsion-free class in Mod–R.

Moreover $\mathcal{C}_P = \{M \in \text{Mod–}R \mid \operatorname{Tor}^1_R(M, \operatorname{Tr}(R/q)) \text{ for all } q \in Q\}$ where $\operatorname{Tr}(R/q)$ denotes the Auslander-Bridger *transpose* of R/q, i.e., the cokernel of the map $f^* = \operatorname{Hom}_R(R, f)$, where $R^m \xrightarrow{f} R \to R/q \to 0$ is a presentation of R/q. Note that $\operatorname{Tr}(R/q)$ has projective dimension ≤ 1 for each $q \in Q$, because $Q \cap \operatorname{Ass}(R) = \emptyset$.

Let $\mathcal{E} = \{\operatorname{Hom}_R(\operatorname{Tr}(R/q), W_{min}) \mid q \in Q\}$. Then $\mathcal{E}$ consists of modules of injective dimension ≤ 1, and $\mathcal{C}_P = {}^{\perp}\mathcal{E}$, whence $\mathcal{C}_P^{\perp}$ consists of modules of injective dimension ≤ 1, too. By Theorem 2.4, $\mathcal{C}_P$ is a 1-cotilting class. □

We will now describe the structure of all cotilting classes over commutative noetherian rings. They are parametrized by the characteristic sequences defined in 3.2 below. Here, for a module M and $i < \omega$, we consider the minimal injective coresolution of M

$$\mathfrak{I}: \quad 0 \to M \to E(M) = I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} \dots$$

and denote by $\Omega^{-i}(M) = \operatorname{Ker}(f_i)$ the *ith cosyzygy* of M in $\mathfrak{I}$.

Definition 3.2. A sequence $\mathcal{P} = (P_0, \dots, P_{n-1})$ of subsets of $\operatorname{Spec}(R)$ of length n is called *characteristic* provided that

(i) P_i is a lower subset of $\operatorname{Spec}(R)$ for each $i < n$,
(ii) $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{n-1}$, and
(iii) $\operatorname{Ass}(\Omega^{-i}(R)) \subseteq P_i$ for all $i < n$.

For each characteristic sequence $\mathcal{P}$, we define a class of modules

$$\mathcal{C}_{\mathcal{P}} = \{M \in \text{Mod–}R \mid \operatorname{Ass}(\Omega^{-i}(M)) \subseteq P_i \text{ for all } i < n\}.$$

Note that a characteristic sequence of length 1 is just (P_0) where P_0 is a lower subset of $\operatorname{Spec}(R)$ containing $\operatorname{Ass}(R)$ (cf. Theorem 3.1 and Example 1.3). The general parametrization goes as follows:

Theorem 3.3 ([1])**.** *Let $\mathcal{P} = (P_0, \dots, P_{n-1})$ be a characteristic sequence. Then $\mathcal{C}_{\mathcal{P}}$ is an n-cotilting class, and the assignments*

$$\mathcal{P} = (P_0, \dots, P_{n-1}) \mapsto \mathcal{C}_{\mathcal{P}}$$

and

$$\mathcal{C} \mapsto (\operatorname{Ass}(\mathcal{C}_0), \dots, \operatorname{Ass}(\mathcal{C}_{n-1}))$$

where $\mathcal{C}_i = {}^{\perp}\{\Omega^{-i}(C)\}$, are mutually inverse bijections between characteristic sequences of length n and n-cotilting classes.

Remark: In the setting of Theorem 3.3, the class $\mathcal{C}_i$ is actually an $(n-i)$-cotilting class.

4. Minimal cotilting modules

Finally, we turn to the question of existence (and structure) of minimal cotilting modules.

Definition 4.1. Let $\mathcal{P} = (P_0, \dots, P_{n-1})$ be a characteristic sequence. Define $P_{-1} = \emptyset$ and $P_n = \mathrm{Spec}(R)$.

For each $0 \leq i \leq n$, let $\mathcal{I}(P_i)$ be the class of all injective modules I with $\mathrm{Ass}(I) \subseteq P_i$. Note that $\mathcal{I}(P_i)$ is a covering class because it is precovering and closed under direct limits - cf. [3].

Let $0 \leq i \leq n$. For a non-empty subset S of $P_i \setminus P_{i-1}$, we let $E_S = \bigoplus_{\mathfrak{p} \in S} E(R/\mathfrak{p})$ and consider the long exact sequence

$$0 \to C_S \to E_0 \overset{\varphi_0}{\to} E_1 \overset{\varphi_1}{\to} \dots \overset{\varphi_{i-2}}{\to} E_{i-1} \overset{\varphi_{i-1}}{\to} E_S \to 0$$

where φ_{i-1} is a $\mathcal{I}(P_{i-1})$-cover of E_S, and for each $0 \leq j < i-1$, μ_j is the inclusion of $K_j = \mathrm{Ker}(\varphi_{j+1})$ into E_{j+1}, $\psi_j : E_j \to K_j$ is a $\mathcal{I}(P_j)$-cover, and $\varphi_j = \mu_j \circ \psi_j$.

For $S = \emptyset$, we let $C_S = 0$.

Theorem 4.2 ([4])**.** *Let $\mathcal{P} = (P_0, \dots, P_{n-1})$ be a characteristic sequence and $\mathcal{C}$ be the corresponding n-cotilting class.*

Then there exists a unique minimal n-cotilting module C_{min} inducing $\mathcal{C}$. In fact,

$$C_{min} \cong C_{S_0} \oplus \dots \oplus C_{S_n}$$

where S_i is the set of all maximal elements in $P_i \setminus P_{i-1}$, for each $i \leq n$.

Final remark: The notions of an n-cotilting class and module are formally dual to the notions of an n-tilting class and module, respectively. Using the Auslander-Bridger transpose one can show that, in the case when R is commutative and noetherian, there is also an explicit duality available, and characteristic sequences parametrize also all n-tilting classes. However, the structure of the n-tilting modules remains open in general. For more details, we refer to [1] and [3].

References

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