ASSOCIATED PRIMES AND INJECTIVE MODULES

JAN TRLIFAJ
PREPARATORY TALK
DELIVERED ON 24 APRIL, 2015

1. Associated primes

Prime ideals are basic objects in classic commutative algebra and algebraic geometry. For example, the irreducibility of an algebraic set A in affine n-space is equivalent to its ideal $I(A)$ in the polynomial ring $k[x_1, \dots, x_n]$ being prime. By Krull's Principal Ideal Theorem, all prime ideals in a commutative noetherian ring R have finite height; the supremum of these heights is the *Krull dimension* of R, [**?**].

Throughout this article, we will assume that R is a commutative noetherian ring.

We will denote by $\mathrm{Spec}(R)$ the *prime spectrum* of R, that is, the set of all prime ideals of R partially ordered by inclusion, and by $\mathrm{mSpec}(R)$ the *maximal spectrum* of R, that is, the subset of $\mathrm{Spec}(R)$ consisting of maximal ideals of R. The notation Mod–R and mod–R will stand for the class of all (R-) modules, and all finitely generated (R-) modules, respectively.

We will be interested in the role of prime ideals in the structure of modules. The simple notion of an associated prime is of key importance here:

Definition 1.1. For a module M, let $\mathrm{Ass}(M) = \{p \in \mathrm{Spec}(R) \mid R/p \text{ embeds into } M\}$. Equivalently, $\mathrm{Ass}(M)$ is the set of all prime ideals that occur as annihilators of elements of M.

The elements of $\mathrm{Ass}(M)$ are called the *associated primes* of M.

Example 1.2. For each $p \in \mathrm{Spec}(R)$, we have $\mathrm{Ass}(R/p) = \{p\}$.

Indeed, if $q \in \mathrm{Spec}(R)$ annihilates a non-zero element $r + p \in R/p$, then $q \subseteq p$ because p is prime, and clearly p annihilates every element of R/p.

Lemma 1.3. $\mathrm{Ass}(M) \neq \emptyset$ *for each non-zero module* M.

Proof. Since R is noetherian, the set of annihilators $\{\mathrm{ann}(x) \mid 0 \neq x \in M\}$ has a maximal element, say $\mathrm{ann}(y)$. If $u.v \in \mathrm{ann}(y)$ and $v \notin \mathrm{ann}(y)$, then $\mathrm{ann}(v.y) = \mathrm{ann}(y)$ by the maximality, so $u \in \mathrm{ann}(y)$. This proves that $\mathrm{ann}(y) \in \mathrm{Spec}(R)$. □

Note that Lemma **??** fails for non-noetherian commutative rings in general. (For example, if $R = \prod_{i\in I} F_i/(\bigoplus_{i\in I} F_i)$, an infinite product of fields F_i modulo their direct sum, then the regular module R has no associated primes.)

The cyclic modules R/p where $p \in \mathrm{Spec}(R)$ can be used to build arbitrary modules by (transfinite) extensions:

Date: September 15, 2016.

Definition 1.4. Let $\mathcal{C}$ be a class of modules. A module M is said to be *$\mathcal{C}$-filtered* (or a *transfinite extension* of the modules in $\mathcal{C}$) provided there is a chain of modules $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ such that $M_0 = 0$, M_α is a submodule of $M_{\alpha+1}$ with $M_{\alpha+1}/M_\alpha$ isomorphic to an element of $\mathcal{C}$ for each $\alpha < \sigma$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and $M_\sigma = M$. The chain $\mathcal{M}$ is called a *$\mathcal{C}$-filtration* of M of length σ.

For example, if $\mathcal{C}$ is the class of all simple modules, then the $\mathcal{C}$-filtered modules are called *semiartinian*, and the finitely $\mathcal{C}$-filtered modules are exactly the modules *of finite length*.

Lemma 1.5. *Each module is $\mathcal{P}$-filtered where $\mathcal{P} = \{R/p \mid p \in \operatorname{Spec}(R)\}$.*

Proof. If $M \neq 0$, then by Lemma **??**, there exists $p \in \operatorname{Spec}(R)$ and a short exact sequence $0 = M_0 \to R/p \overset{\mu}{\to} M \to M' \to 0$. If $M' \neq 0$, we let $M_1 = \mu(R/p)$ and apply Lemma **??** to M/M_1 in order to obtain a $p' \in \operatorname{Spec}(R)$ and a short exact sequence $0 \to R/p' \to M/M_1 \to M'' \to 0$, etc. Proceeding similarly in non-limit steps (and taking unions in the limit ones), we eventually arrive at an ordinal σ such that $M_\sigma = M$. Then $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ is the desired $\mathcal{P}$-filtration of M. $\square$

Note that $\mathcal{P}$-filtrations are not unique in general: for example, if R is a domain, then $(0, R)$ is a $\mathcal{P}$-filtration of the regular module R, but for each $m \in \operatorname{mSpec}(R)$, any $\mathcal{P}$-filtration of m can be prolonged, by adding R as the last term, to a $\mathcal{P}$-filtration of R.

We also mention an easy observation concerning associated primes:

Lemma 1.6. *For each short exact sequence of modules, $0 \to M \to N \to P \to 0$, we have $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(N) \subseteq \operatorname{Ass}(M) \cup \operatorname{Ass}(P)$.*

Since finitely generated modules have finite $\mathcal{P}$-filtrations, we immediately obtain

Corollary 1.7. *$\operatorname{Ass}(M)$ is finite for each finitely generated module M.*

Another immediate corollary will be useful later on for understanding torsion-free classes of modules:

Corollary 1.8. *Let $P \subseteq \operatorname{Spec}(R)$ and let $\mathcal{Q} = \{R/p \mid p \in P\}$ be the corresponding subset of $\mathcal{P}$.*

Then the class $\mathcal{C}_P = \{M \in \text{Mod–}R \mid \operatorname{Ass}(M) \subseteq P\}$ is closed under submodules and extensions. Moreover, $\mathcal{C}_P$ contains all $\mathcal{Q}$-filtered modules.

We finish our exposition on associated primes by a less well-known variant of Lemma **??** due to Hochster:

Lemma 1.9. *Let M be a finitely generated module. Let $\mathcal{C} = \{N \in \text{mod–}R \mid \exists p \in \operatorname{Ass}(M) : N$ embeds into $R/p\}$. Then M is $\mathcal{C}$-filtered.*

2. Matlis theory of indecomposable injective modules

Now we briefly review the structure and basic properties of injective modules over commutative noetherian rings. For details, see [**?**]. The basic theorem on their (unique) decomposition into indecomposable summands holds even in the one-sided noetherian setting, but the structure of indecomposable injectives is particular to the commutative case:

Theorem 2.1. *Each injective module is (uniquely) a direct sum of copies of the indecomposable injective modules $E(R/p)$ (= injective envelopes of the modules R/p) for some $p \in \operatorname{Spec}(R)$.*

Indecomposable injectives can easily be recognized by their associated primes:

Example 2.2. $\mathrm{Ass}(E(R/p)) = \{p\}$ for each $p \in \mathrm{Spec}(R)$.

Indeed, if $q \in \mathrm{Spec}(R)$ and R/q embeds into $E(R/p)$, then its intersection with R/p is non-zero, in contradiction with Example **??**.

The following important facts concerning the structure of $E(R/p)$ for $p \in \mathrm{Spec}(R)$ are due to Matlis:

Lemma 2.3. *(1) Each $x \in E(R/p)$ is annihilated by p^n for some finite n. However, if $r \in R \setminus p$, then the multiplication by r is an automorphism of $E(R/p)$.*

(2) Denote by R_p the localization of R at p, and let $k(p) = R_p/p_p$. Then $E(R/p)$ is also an indecomposable injective R_p-module, namely the injective envelope of the simple R_p-module $k(p)$. There are R-module inclusions $R/p \subseteq k(p) \subseteq E(R/p)$, and the R_p-module $E(R/p)$ is $\{k(p)\}$-filtered and artinian.

Consequently, homomorphisms between indecomposable injective modules respect the partial order on $\mathrm{Spec}(R)$:

Lemma 2.4. *Let $p, q \in \mathrm{Spec}(R)$. Then $Hom_R(E(R/p), E(R/q)) \neq 0$, if and only if $p \subseteq q$.*

Proof. If $p \subseteq q$, then the epimorphism $R/p \to R/q$ extends to a non-zero homomorphism of the respective injective envelopes.

Conversely, if $r \in p \setminus q$ and $f \in \mathrm{Hom}_R(E(R/p), E(R/q))$ is such that $f(x) \neq 0$ for some $x \in E(R/p)$, then $r^n x = 0$ for some finite $n > 0$ by Lemma **??**(1), so $r^n f(x) = f(r^n x) = 0$. However, the multiplication by r is an automorphism of $E(R/q)$, again by Lemma **??**(1), so $r^n f(x) \neq 0$, a contradiction. □

Injectivity is closely related to the property of being a cogenerator in the sense of the following definition:

Definition 2.5. An module I is a *cogenerator* in case each module embeds in a (possibly infinite) direct product of copies of I.

The module $C_{min} = \bigoplus_{m \in mSpecR} E(R/m)$ is injective and a cogenerator. Moreover, it is isomorphic to a direct summand in any injective cogenerator, whence C_{min} is called the *minimal injective cogenerator*.

3. Applications to torsion pairs of modules

In order to present a classification of cotilting classes over commutative noetherian rings later on, we will recall some classic facts on the structure of torsion pairs and classes of modules in that setting.

Definition 3.1. (1) A pair of classes of modules $\mathfrak{T} = (\mathcal{T}, \mathcal{F})$ is a *torsion pair*, if $\mathcal{T} = \{T \mid \mathrm{Hom}_R(T, F) = 0$ for each $F \in \mathcal{F}\}$ and $\mathcal{F} = \{F \mid \mathrm{Hom}_R(T, F) = 0$ for each $T \in \mathcal{T}\}$. $\mathcal{T}$ is then called the *torsion class*, and $\mathcal{F}$ the *torsion-free class* of $\mathfrak{T}$.

Note: These definitions make sense both in mod–R, and in Mod–R.

(2) A torsion pair $\mathfrak{T}$ in Mod–R is *hereditary*, if $\mathcal{T}$ is closed under submodules, or equivalently, $\mathcal{F}$ is closed under injective envelopes.

We finish by recalling how the associated primes mediate a correspondence between torsion-free classes and *lower* subsets of $\operatorname{Spec}(R)$ (i.e., the subsets P such that if $p \in P$ and $q \subseteq p$, then $q \in \operatorname{Spec}(R)$):

Theorem 3.2 (Gabriel)**.** *(1) Torsion-free classes $\mathcal{C}$ in* mod–R *correspond 1-1 to lower subsets P of* $\operatorname{Spec}(R)$*, via the inverse assignments*
$P \mapsto \{M \in \text{mod–}R \mid \operatorname{Ass}(M) \subseteq P\}$ *and*
$\mathcal{C} \mapsto \{p \in \operatorname{Spec}(R) \mid p \in \operatorname{Ass}(M) \textit{ for some } M \in \mathcal{C}\}$.

(2) Hereditary torsion pairs $\mathfrak{T}$ in Mod–R *correspond 1-1 to lower subsets P of* $\operatorname{Spec}(R)$*, via the inverse assignments*
$P \mapsto \mathfrak{T}_P = (\mathcal{T}_P, \mathcal{C}_P)$ *where* $\mathcal{C}_P = \{M \in \text{Mod–}R \mid \operatorname{Ass}(M) \subseteq P\}$*, and*
$\mathfrak{T} = (\mathcal{T}, \mathcal{C}) \mapsto P = \{p \in \operatorname{Spec}(R) \mid p \in \operatorname{Ass}(M) \textit{ for some } M \in \mathcal{C}\}$.

Remark: While hereditary torsion pairs are easily classified by Theorem **??**(2), the classification of all torsion pairs is hopeless in general - e.g., there is a proper class of torsion pairs of abelian groups.

References

[1] D. Eisenbud, Commutative Algebra (with a view toward algebraic geometry). Springer, GTM **150**, 1995.

[2] H. Matsumura, Commutative Ring Theory, Cambridge, CSAM **8**, 1994.