ASYMPTOTIC PRIME DIVISORS OVER COMPLETE INTERSECTION RINGS

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1. Introduction

Let A be a commutative Noetherian ring, I an ideal of A, and M a finitely generated A-module. Brodmann [2] proved that the set of associated prime ideals $\operatorname{Ass}_A(M/I^nM)$ is independent of n for all sufficiently large n. Thereafter, L. Melkersson and P. Schenzel generalized Brodmann's result in [5, Theorem 1] by proving that

$$\operatorname{Ass}_A\left(\operatorname{Tor}_i^A(M, A/I^n)\right)$$

is independent of n for all large n and for a fixed $i \geqslant 0$.

Later, D. Katz and E. West proved the above result in a more general way [4, 3.5]; if N is a finitely generated A-module, then for a fixed $i \geqslant 0$, the sets

$$\operatorname{Ass}_A\left(\operatorname{Tor}_i^A(M, N/I^nN)\right) \quad \text{and} \quad \operatorname{Ass}_A\left(\operatorname{Ext}_A^i(M, N/I^nN)\right)$$

are stable for all large n. So, in particular, for a fixed $i \geqslant 0$,

$$\bigcup_{n\geqslant 0} \operatorname{Ass}_A\left(\operatorname{Ext}_A^i(M, N/I^nN)\right)$$

is a finite set. In this context, Tony J. Puthenpurakal [6, page 368] raised a question about what happens if we vary i ($\geqslant 0$) also? More precisely,

$$(\dagger) \qquad \text{is the set } \bigcup_{i\geqslant 0}\bigcup_{n\geqslant 0} \operatorname{Ass}_A\left(\operatorname{Ext}_A^i(M, N/I^nN)\right) \text{ finite?}$$

The motivation for the question (†) came from the following two questions. They were raised by W. Vasconcelos [7, 3.5] and Melkersson and Schenzel [5, page 936] respectively.

$$(1) \qquad \text{Is the set } \bigcup_{i\geqslant 0} \operatorname{Ass}_A\left(\operatorname{Ext}_A^i(M, A)\right) \text{ finite?}$$

$$(2) \qquad \text{Is the set } \bigcup_{i\geqslant 0}\bigcup_{n\geqslant 1} \operatorname{Ass}_A\left(\operatorname{Tor}_i^A(M, A/I^n)\right) \text{ finite?}$$

Recently, Tony J. Puthenpurakal [6, Theorem 5.1] proved that if A is a local complete intersection ring and $\mathcal{N} = \bigoplus_{n\geqslant 0} N_n$ is a finitely generated graded module over the Rees ring $\mathscr{R}(I)$, then

$$\bigcup_{i\geqslant 0}\bigcup_{n\geqslant 0} \operatorname{Ass}_A\left(\operatorname{Ext}_A^i(M, N_n)\right)$$

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[1]This is a joint work with Prof. Tony J. Puthenpurakal.

is a finite set. Moreover, he proved that there exists $i_0, n_0 \geqslant 0$ such that

$$\operatorname{Ass}_A\left(\operatorname{Ext}_A^{2i}(M, N_n)\right) = \operatorname{Ass}_A\left(\operatorname{Ext}_A^{2i_0}(M, N_{n_0})\right),$$
$$\operatorname{Ass}_A\left(\operatorname{Ext}_A^{2i+1}(M, N_n)\right) = \operatorname{Ass}_A\left(\operatorname{Ext}_A^{2i_0+1}(M, N_{n_0})\right)$$

for all $i \geqslant i_0$ and $n \geqslant n_0$. In particular, if N is a finitely generated A-module, then $\mathcal{N}$ can be taken as $\bigoplus_{n\geqslant 0}(I^n N)$ or $\bigoplus_{n\geqslant 0}(I^n N/I^{n+1}N)$. In the present study, we prove that the question (†) has an affirmative answer for a local complete intersection ring. We also analyze the stability of the sets of associated prime ideals which occurs periodically after certain stage.

Let A be a local complete intersection ring. Let M, N be two finitely generated A-modules and I an ideal of A. The complexity of a pair of modules was introduced in [1] by Avramov and Buchweitz. The complexity of the pair of A-modules (M, N) is defined to be the number

$$\operatorname{cx}_A(M, N) = \inf\left\{ b \in \mathbb{N} \;\middle|\; \limsup_{n\to\infty} \frac{\mu(\operatorname{Ext}_A^n(M, N))}{n^{b-1}} < \infty \right\},$$

where $\mu(D)$ denote the minimal number of generators of a finitely generated A-module D. In [6, Theorem 7.1], Tony J. Puthenpurakal proved that $\operatorname{cx}_A(M, I^j N)$ is constant for all $j \gg 0$. We prove that

$$(\dagger\dagger) \qquad \operatorname{cx}_A(M, N/I^j N) \quad \text{is constant for all } j \gg 0.$$

2. Main results

Throughout this section, let A be a local complete intersection ring. Let M, N be two finitely generated A-modules and I an ideal of A. We prove the following results on associate primes.

Theorem 2.1. $\displaystyle\bigcup_{i\geqslant 0}\bigcup_{n\geqslant 0} \operatorname{Ass}_A\left(\operatorname{Ext}_A^i\left(M, \frac{N}{I^n N}\right)\right)$ *is a finite set.*

Theorem 2.2. *There exists $i_0, n_0 \in \mathbb{N}$ such that for all $i \geqslant i_0$ and $n \geqslant n_0$, we have*

$$\operatorname{Ass}_A\left(\operatorname{Ext}_A^{2i}\left(M, \frac{N}{I^n N}\right)\right) = \operatorname{Ass}_A\left(\operatorname{Ext}_A^{2i_0}\left(M, \frac{N}{I^{n_0} N}\right)\right),$$
$$\operatorname{Ass}_A\left(\operatorname{Ext}_A^{2i+1}\left(M, \frac{N}{I^n N}\right)\right) = \operatorname{Ass}_A\left(\operatorname{Ext}_A^{2i_0+1}\left(M, \frac{N}{I^{n_0} N}\right)\right).$$

Recall that a Noetherian local ring A is said to be a *complete intersection ring* if its completion $\widehat{A} = Q/(\mathbf{f})$, where Q is a complete regular local ring (RLR) and $\mathbf{f} = f_1, \ldots, f_c$ is a Q-regular sequence. We prove our results for rings of types $Q/(\mathbf{f})$ (where Q is a complete RLR and $\mathbf{f}$ is a Q-regular sequence). Once we have the results for these types of rings, i.e., for $\widehat{A}$, then we will have the results for A by the following lemma.

Lemma 2.3. [6, 5.6.(b)] *For a finitely generated module D over a Noetherian local ring A, we have*

$$\operatorname{Ass}_A(D) = \{P \cap A : P \in \operatorname{Ass}_{\widehat{A}}(D \otimes_A \widehat{A})\}.$$

Now we give

Proof of Theorem 2.1. We may assume that $A = Q/(\mathbf{f})$, where Q is a complete RLR and $\mathbf{f} = f_1, \ldots, f_c$ is a Q-regular sequence. For a fixed $n \geqslant 0$, consider the short exact sequence of A-modules:

$$0 \longrightarrow I^nN/I^{n+1}N \longrightarrow N/I^{n+1}N \longrightarrow N/I^nN \longrightarrow 0.$$

Taking direct sum over $n \geqslant 0$, and setting $\mathcal{L} := \bigoplus_{n\geqslant 0}(N/I^{n+1}N)$, we have a short exact sequence of graded $\mathscr{R}(I)$-modules:

$$0 \longrightarrow \operatorname{gr}_I(N) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(-1) \longrightarrow 0,$$

which induces an exact sequence of $\mathscr{R}(I)$-modules for each $i \geqslant 0$:

$$\operatorname{Ext}^i_A(M, \operatorname{gr}_I(N)) \longrightarrow \operatorname{Ext}^i_A(M, \mathcal{L}) \longrightarrow \operatorname{Ext}^i_A(M, \mathcal{L}(-1)).$$

Taking direct sum over $i \geqslant 0$ and using the naturality of the Eisenbud operators t_j (cf. [3, Section 1]), we have an exact sequence of $\mathscr{S} = \mathscr{R}(I)[t_1, \ldots, t_c]$-modules:

$$\bigoplus_{i,n\geqslant 0} \operatorname{Ext}^i_A\left(M, \frac{I^nN}{I^{n+1}N}\right) \xrightarrow{\Phi} \bigoplus_{i,n\geqslant 0} V_{i,n} \xrightarrow{\Psi} \bigoplus_{i,n\geqslant 0} V_{i,n-1},$$

where $V_{i,n} := \operatorname{Ext}^i_A(M, N/I^{n+1}N)$ for each $i \geqslant 0, n \geqslant -1$. Let

$$U = \bigoplus_{i,n\geqslant 0} U_{i,n} := \operatorname{Image}(\Phi).$$

Then for each $i, n \geqslant 0$, considering the exact sequence of A-modules:

$$0 \to U_{i,n} \to V_{i,n} \to V_{i,n-1},$$

we have

$$\begin{aligned}
\operatorname{Ass}_A(V_{i,n}) &\subseteq \operatorname{Ass}_A(U_{i,n}) \cup \operatorname{Ass}_A(V_{i,n-1}) \\
&\subseteq \operatorname{Ass}_A(U_{i,n}) \cup \operatorname{Ass}_A(U_{i,n-1}) \cup \operatorname{Ass}_A(V_{i,n-2}) \\
&\vdots \\
&\subseteq \bigcup_{0\leqslant j\leqslant n} \operatorname{Ass}_A(U_{i,j}). \quad [\text{Since } \operatorname{Ass}_A(V_{i,-1}) = \phi \text{ for each } i \geqslant 0].
\end{aligned}$$

Taking union over $i, n \geqslant 0$, we have

$$\bigcup_{i,n\geqslant 0} \operatorname{Ass}_A(V_{i,n}) \subseteq \bigcup_{i,n\geqslant 0} \operatorname{Ass}_A(U_{i,n}). \tag{2.1}$$

Since $\operatorname{gr}_I(N)$ is a finitely generated graded $\mathscr{R}(I)$-module, by [6, Theorem 1.1],

$$\bigoplus_{i,n\geqslant 0} \operatorname{Ext}^i_A\left(M, \frac{I^nN}{I^{n+1}N}\right)$$

is a finitely generated bigraded $\mathscr{S}$-module, and hence U is a finitely generated bigraded $\mathscr{S}$-module. Therefore by [8, Lemma 3.2],

$$\bigcup_{i,n\geqslant 0} \operatorname{Ass}_A(U_{i,n}) \quad \text{is a finite set.} \tag{2.2}$$

The result follows from (2.1) and (2.2). □

To prove Theorem 2.2, we use the following lemma:

Lemma 2.4. *Let* $(Q,\mathfrak{n})$ *be a Noetherian local ring with residue field* k*, and let* $\mathbf{f} = f_1,\ldots,f_c$ *be a* Q*-regular sequence. Set* $A = Q/(\mathbf{f})$*. Let* M,N *be two finitely generated* A*-modules with* $\operatorname{projdim}_Q(M)$ *finite, and* I *an ideal of* A*. Then*

$$\lambda_A\left(\operatorname{Hom}_A\left(k,\operatorname{Ext}_A^{2i}(M,N/I^nN)\right)\right) \quad \text{and} \quad \lambda_A\left(\operatorname{Hom}_A\left(k,\operatorname{Ext}_A^{2i+1}(M,N/I^nN)\right)\right)$$

are given by polynomials in i,n *with rational coefficients for all large* (i,n)*.*

We prove the following result on complexity.

Theorem 2.5. *Let* A *be a local complete intersection ring. Let* M*,* N *be two finitely generated* A*-modules and* I *an ideal of* A*. Then*

$$\operatorname{cx}_A(M,N/I^jN) \quad \text{is constant for all } j \gg 0.$$

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