INTRODUCTION TO EXT

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1. Introduction

Let R be a Commutative ring. For any two R-modules M and N, $\operatorname{Hom}_R(M, N)$ denotes the collection of all R-module homomorphisms from M to N. Since R is a commutative ring, one can check that $\operatorname{Hom}_R(M, N)$ is an R-module as well. We denote the *category* of all R-modules and all R-module homomorphisms by $\operatorname{Mod}(R)$. Let N be an R-module. Define a *functor* $\operatorname{Hom}_R(-, N) : \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ by mapping each R-module M to $\operatorname{Hom}_R(M, N)$, and each R-module homomorphism $f : M_1 \to M_2$ to

$$\begin{aligned}\operatorname{Hom}_R(f, N) : \operatorname{Hom}_R(M_2, N) &\longrightarrow \operatorname{Hom}_R(M_1, N).\\ g &\longmapsto g \circ f\end{aligned}$$

Clearly, $\operatorname{Hom}_R(f, N)$ is an R-module homomorphism. The functor $\operatorname{Hom}_R(-, N)$ is *left exact*, i.e., it takes an exact sequence $0 \to M' \to M \to M'' \to 0$ into an exact sequence $0 \to \operatorname{Hom}_R(M'', N) \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M', N)$. Recall that $\operatorname{Hom}_R(-, N)$ is exact if and only if N is an *injective* R-module. So it is not necessarily true that when we apply $\operatorname{Hom}_R(-, N)$ on a short exact sequence, then we get another short exact sequence. But in this process we get a long exact sequence where some new modules will come into the picture. Those modules are called Ext modules.

2. Complexes

In this section, we discuss complexes and their homology and cohomology modules.

By a *chain complex*, we mean a sequence

$$\mathcal{C} : \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$

of R-modules and R-module homomorphisms such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. Since $\operatorname{Image}(d_{n+1}) \subseteq \operatorname{Ker}(d_n)$, we define $H_n(\mathcal{C}) := \operatorname{Ker}(d_n)/\operatorname{Image}(d_{n+1})$ to be the n^{th} *homology module* of $\mathcal{C}$. If $H_n(\mathcal{C}) = 0$ for all n, then we say $\mathcal{C}$ is an *exact sequence.*

By a *cochain complex*, we mean a sequence

$$\mathcal{D} : \quad \cdots \to C_{n-1} \xrightarrow{d_{n-1}} C_n \xrightarrow{d_n} C_{n+1} \to \cdots$$

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of R-modules and R-module homomorphisms such that $d_n \circ d_{n-1} = 0$ for all $n \in \mathbb{Z}$. Since $\operatorname{Image}(d_{n-1}) \subseteq \operatorname{Ker}(d_n)$, we define $H^n(\mathcal{C}) := \operatorname{Ker}(d_n)/\operatorname{Image}(d_{n-1})$ to be the n^{th} *cohomology module* of $\mathcal{D}$.

A *morphism* $f : \mathcal{C} \to \mathcal{C}'$ of complexes is a family $f = (f_n)_{n\in\mathbb{Z}}$ of R-module homomorphisms $f_n : C_n \to C'_n$ satisfying $d'_n \circ f_n = f_{n-1} \circ d_n$ for all $n \in \mathbb{Z}$, or in other words we have a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow{d'_{n+1}} & C'_n & \xrightarrow{d'_n} & C'_{n-1} & \longrightarrow & \cdots .
\end{array}$$

It is easy to see that each morphism $f : \mathcal{C} \to \mathcal{C}'$ of complexes induces R-module homomorphisms $H_n(f) : H_n(\mathcal{C}) \to H_n(\mathcal{C}')$ for all $n \in \mathbb{Z}$ between the homology modules.

Let $f, g : \mathcal{C} \to \mathcal{C}'$ be two morphisms of complexes. We say that f and g are *homotopic* if for each n there is a homomorphism $h_n : C_n \to C'_{n+1}$ such that $f_n - g_n = d'_{n+1} \circ h_n + h_{n-1} \circ d_n$. By $f \sim g$, we mean f and g are homotopic. If $f \sim g$, then they induce the same homomorphisms $H_n(\mathcal{C}) \to H_n(\mathcal{C}')$ on homology modules.

Two complexes $\mathcal{C}$ and $\mathcal{C}'$ are said to be *homotopy equivalent* if there exist morphisms $f : \mathcal{C} \to \mathcal{C}'$ and $g : \mathcal{C}' \to \mathcal{C}$ such that $g \circ f \sim \mathrm{id}_{\mathcal{C}}$ and $f \circ g \sim \mathrm{id}_{\mathcal{C}'}$, where $\mathrm{id}_{\mathcal{C}}$ denotes the identity morphism $\mathcal{C} \to \mathcal{C}$. Homotopy equivalent complexes have the same homology.

A sequence of complexes $0 \to \mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{g} \mathcal{C}'' \to 0$ is said to be *exact* if $0 \to C'_n \xrightarrow{f_n} C_n \xrightarrow{g_n} C''_n \to 0$ is exact for every $n \in \mathbb{Z}$. We refer the reader to [**?**, Theorem 1.3.1] for a proof of the following well-known result.

Theorem 2.1. *Let $0 \to \mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{g} \mathcal{C}'' \to 0$ be a short exact sequence of chain complexes. Then there are natural homomorphisms $\delta_n : H_n(\mathcal{C}'') \to H_{n-1}(\mathcal{C}')$, called connecting homomorphisms, such that the following sequence is exact:*

$$\begin{array}{l}
\cdots \xrightarrow{H_{n+1}(g)} H_{n+1}(\mathcal{C}'') \\
\xrightarrow{\delta_{n+1}} H_n(\mathcal{C}') \xrightarrow{H_n(f)} H_n(\mathcal{C}) \xrightarrow{H_n(g)} H_n(\mathcal{C}'') \\
\xrightarrow{\delta_n} H_{n-1}(\mathcal{C}') \xrightarrow{H_{n-1}(f)} \cdots .
\end{array}$$

Similarly, if $0 \to \mathcal{D}' \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{D}'' \to 0$ is a short exact sequence of cochain complexes, then there are connecting homomorphisms $\delta^n : H^n(\mathcal{D}'') \to H^{n+1}(\mathcal{D}')$ and a long exact sequence:

$$\begin{array}{l}
\cdots \xrightarrow{H^{n-1}(g)} H^{n-1}(\mathcal{D}'') \\
\xrightarrow{\delta^{n-1}} H^n(\mathcal{D}') \xrightarrow{H^n(f)} H^n(\mathcal{D}) \xrightarrow{H^n(g)} H^n(\mathcal{D}'') \\
\xrightarrow{\delta^n} H^{n+1}(\mathcal{D}') \xrightarrow{H^{n+1}(f)} \cdots .
\end{array}$$

□

An R-module P is said to be a *projective module* if it satisfies the following universal lifting property: Given a surjection $g : N \to N''$ and a homomorphism

$u : P \to N''$, there is a homomorphism $v : P \to N$ such that $u = g \circ v$. It is easy to see that an R-module is projective if and only if it is a direct summand of a free R-module. So any R-module can be written as a quotient of a projective module. Using this property, for any R-module M, we can get a chain complex

$$\mathcal{P} : \ \cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

of projective R-modules P_i such that

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

is exact. The chain complex $\mathcal{P}$ is called a *projective resolution* of M. If R is Noetherian and M is finitely generated, then we can take each P_n to be a free module of finite rank. We write $\mathcal{P} \xrightarrow{\epsilon} M$ to denote a projective resolution $\mathcal{P}$ of M. Let us recall the following result from [**?**, Theorem 2.2.6].

Theorem 2.2. *Let $\mathcal{P} \xrightarrow{\epsilon} M$ be a projective resolution of an R-module M and let $f' : M \to N$ be an R-module homomorphism. Then for every resolution $\mathcal{Q} \xrightarrow{\eta} N$ of N, there is a morphism $f : \mathcal{P} \to \mathcal{Q}$ of complexes lifting f' in the sense that the following diagram is commutative:*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f'} & & \\ \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \xrightarrow{\eta} & N & \longrightarrow & 0. \end{array}$$

The morphism $f : \mathcal{P} \to \mathcal{Q}$ is unique up to homotopy equivalence. □

As a corollary we obtain the following result.

Corollary 2.3. *Any two projective resolutions of a module are homotopy equivalent.*

3. The functors Ext

We start this section by giving the construction of Ext modules by considering projective resolutions of modules.

Definition 3.1. Let M and N be two R-modules. Consider a projective resolution

$$\mathcal{P} : \ \cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

of M. Apply the functor $\operatorname{Hom}_R(-, N)$ on $\mathcal{P}$ to get the following cochain complex:

$$\operatorname{Hom}_R(\mathcal{P}, N) : 0 \xrightarrow{\Phi_{-1}} \operatorname{Hom}_R(P_0, N) \xrightarrow{\Phi_0} \operatorname{Hom}_R(P_1, N) \xrightarrow{\Phi_1} \operatorname{Hom}_R(P_2, N) \longrightarrow \cdots .$$

The n^{th} *Ext module* of the pair (M, N) is defined to be

$$\operatorname{Ext}^n_R(M, N) := H^n(\operatorname{Hom}_R(\mathcal{P}, N)) = \operatorname{Ker}(\Phi_n) / \operatorname{Image}(\Phi_{n-1}).$$

Remark 3.2. $\operatorname{Ext}^n_R(M, N) = 0$ for every negative integer n. $\operatorname{Ext}^n_R(M, N)$ does not depend on the choice of projective resolution of M as any two projective resolutions of M are homotopy equivalent.

Now we give a few properties of Ext modules.

E 1. *For a fixed R-module N, $\operatorname{Ext}^n_R(-, N) : \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ is a contravariant R-linear functor for every integer n.*

Proof. $\text{Ext}_R^n(-, N)$ maps an R-module M to $\text{Ext}_R^n(M, N)$. For an R-module homomorphism $f : M_1 \to M_2$, we have to find the homomorphisms $\text{Ext}_R^n(f, N) : \text{Ext}_R^n(M_2, N) \to \text{Ext}_R^n(M_1, N)$. Consider two projective resolutions $\mathcal{P} \xrightarrow{\epsilon} M_1$ and $\mathcal{Q} \xrightarrow{\eta} M_2$ of M_1 and M_2 respectively. By Theorem **??**, there is a morphism $\Phi : \mathcal{P} \to \mathcal{Q}$ (unique up to homotopy equivalence) lifting f. Applying $\text{Hom}_R(-, N)$ on Φ, we get a morphism $\text{Hom}_R(\Phi, N) : \text{Hom}_R(\mathcal{Q}, N) \to \text{Hom}_R(\mathcal{P}, N)$ of cochain complexes which induces the homomorphisms

$$\text{Ext}_R^n(f, N) : \text{Ext}_R^n(M_2, N) \to \text{Ext}_R^n(M_1, N)$$

on cohomology modules, i.e., $\text{Ext}_R^n(f, N) := H^n(\text{Hom}_R(\Phi, N))$. Now one can check that $\text{Ext}_R^n(-, N)$ is actually an R-linear functor. □

E 2. *Two functors* $\text{Hom}_R(-, N)$ *and* $\text{Ext}_R^0(-, N)$ *are isomorphic.*

Proof. Consider a projective presentation $P_1 \longrightarrow P_0 \xrightarrow{\epsilon} M \to 0$ of an R-module M. Since $\text{Hom}_R(-, N)$ is left exact, we obtain the following exact sequence

$$0 \longrightarrow \text{Hom}_R(M, N) \longrightarrow \text{Hom}_R(P_0, N) \xrightarrow{D_0} \text{Hom}_R(P_1, N),$$

which gives $\text{Hom}_R(M, N) \cong \text{Ker}(D_0) = \text{Ext}_R^0(M, N)$. We omit the complete proof here because it is quite technical. □

E 3. *If* M *is a projective* R*-module, or if* N *is an injective* R*-module, then*

$$\text{Ext}_R^n(M, N) = 0 \quad \textit{for all } n \geqslant 1.$$

Proof. If M is a projective R-module, then we get the result by computing the Ext modules using the projective resolution $\cdots \to 0 \to M \to 0$ of M. If N is an injective R-module, then we can use the fact that $\text{Hom}_R(-, N)$ is an exact functor to get the required result. □

E 4. *For each short exact sequence* $0 \to L' \to L \to L'' \to 0$ *of* R*-modules, there exists a long exact sequence*

$$\begin{aligned} 0 &\longrightarrow \text{Hom}_R(L'', N) \longrightarrow \text{Hom}_R(L, N) \longrightarrow \text{Hom}_R(L', N) \\ &\longrightarrow \text{Ext}_R^1(L'', N) \longrightarrow \text{Ext}_R^1(L, N) \longrightarrow \text{Ext}_R^1(L', N) \\ &\longrightarrow \text{Ext}_R^2(L'', N) \longrightarrow \cdots . \end{aligned}$$

Proof. Consider two projective resolutions $\mathcal{P}' \to L'$ and $\mathcal{P}'' \to L''$ of L' and L'' respectively. Then, by Horseshoe Lemma ([**?**, 2.2.8]), we obtain a projective resolution $\mathcal{P} \to L$ of L which yields the following split short exact sequence

$$0 \longrightarrow \mathcal{P}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}'' \longrightarrow 0 \tag{3.1}$$

of chain complexes. Applying $\text{Hom}_R(-, N)$ on (**??**), we get the following short exact sequence of cochain complexes:

$$0 \longrightarrow \text{Hom}_R(\mathcal{P}'', N) \longrightarrow \text{Hom}_R(\mathcal{P}, N) \longrightarrow \text{Hom}_R(\mathcal{P}', N) \longrightarrow 0. \tag{3.2}$$

Then we get the desired long exact sequence by Theorem **??**. □

References

[Mat] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1986.
[Wei] C. A. Weibel, *An introduction to homological algebra*, Cambridge University Press, Cambridge, 1994.