TRIBHUVAN UNIVERSITY

Faculty of Humanities and Social Sciences

B. A. Mathematics Courses

Micro Syllabus

Prepared by

Central Department of Mathematics
Tribhuvan University, Kirtipur

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1 Math 421 Detailed Course of Calculus

Course Title: Calculus (Compulsory)	**Full Marks:** 100
Course No.: Math 421	**Pass Mark:** 40
Level: B.A.	**Year:** I
Nature of Course: Theory	**Periods:** 9 Lectures Hrs/Week

Course Objectives: This course is designed for the first year of Four Years B.A. Program as a compulsory subject in mathematics. The main aim of this course is to provide knowledge of calculus and use of calculus in different geometrical aspects and vectors. The objective of this course is to acquaint students with the basic concepts, and to build good knowledge base in both differential and integral calculus.

Detailed Course Contents:

Unit 1 Functions, Limits and Continuity [15 Lecture Hrs]

Functions: domain and range, Graph of functions, Vertical line test for a function, Piecewise defined functions, Increasing and decreasing functions, Even and odd functions, Symmetry of a functions, Types of functions (Algebraic functions and Transcendental functions), Combining functions, Shifting and scaling functions (sum, product, difference and quotient of functions, composite functions, vertical and horizontal shifting, scaling and reflecting of graph of functions)
Exercise: 1.1: 1 to 6, 23, 25 to 27, 37 to 40, 47 to 54.
Exercise- 1.2: 1 to 10,
Rate of change and tangent to curves, Examples 1,2,3, Limit of a function and Limit laws, Sandwich theorem, Examples 1 to 11,
Exercise 2.2: 39 to 42, 63
Precise definition of limits, Example 1, Definition of limit, Examples 2,4,6; One sided limits , Example 1, Theorem 6;
Continuity at a point, Properties of continuous functions, Example 12,
Exercise- 2.5: 13, 29, 30, 43, 44, 47
Asymptotes (Horizontal, vertical and slant asymptotes) and related problems; Examples 1,2,3,4,9, 15,16. Related problems.

Unit 2 Differentiation [15 Lecture Hrs]

Tangents and derivative at a point; Example 1; The derivative as a function, Calculating derivative from the definition, Examples 1,2,4;
Theorem 1;
Differentiation rule; Derivative as a rate of change, Examples 1,3,5; The chain rule, Examples 1,2; Implicit differentiation, Examples 2,3; Linearization and differentials, Examples 1,2,3,4;
Higher order derivatives and Leibnitz theorem,
Exercise 3(I) and 3(II) and all the solved examples [From the book "Differential Calculus "By M.B. Singh and B.C. Bajracharya , Sukunda Pustak Bhawan]
Related problems.

Unit 3 Application of Derivatives [15 Lecture Hrs]

Absolute extreme values and local extreme values, Critical point, Examples 1,2 ; monotonic functions, Example 1, First derivative test for local extrema, Example 2,
Exercise- 4.3: 1,3,5,7;
Concavity (The Second derivative test), Examples:1,2,3,4; Second derivative test for local extrema, Examples 7,9;
Exercise- 4.4: 9,11, 13, 15,17;
Indeterminate forms and L'Hospital's rule, Theorem-5(Without proof), Examples 1 to 8;
Exercise 7.5: 1,3,5,7,13,19,23,27;
Mean value theorem (Without proof), Theorem-4, Theorem -3, Rolle's theorem(Without proof); Examples 1,2,3
Exercise 4.2: 1 to 12, 14, 19, 21, 25;
Taylor and Maclaurin Series(Without proof), Worked Out Examples: 1,2,3,4,5,6,7,8,9,
Exercise 4(II)[From the book "Differential Calculus' by M.B. Singh and B.C. Bajracharya]
Related problems.

Unit 4 Integrals [15 Lecture Hrs]

Antiderivative, Basic Integration formulae, substitution method, integration by parts and related problems, estimating with finite sums, Example 1; Sigma notation and limit of finite sum, Examples 1,2,3,4; Riemann

sum, Definite integral, Properties of definite integral, Examples 1,2,3,4; Fundamental theorem of integral calculus, Mean value theorem of definite integral (statement only), Fundamental theorem of calculus part I, Example 2, Fundamental theorem of calculus part II, Examples 3,8;
Exercise 5.4: 1, 5, 7, 21, 27, 39, 41
Improper integrals, Improper integral of type I and II; Examples 1,2,3,4,5;
Exercise 8.7: 1 to 10
Numerical Integration, Trapezoidal approximation and Simpson's rule (without derivation), Example 2, Error analysis,
Theorem-1(without proof),
Examples 1, 4
Related problems.

Unit 5 Application of Definite Integrals [15 Lecture Hrs]
Area between curves, Examples 4,5,6;
Exercise-5.6: 41,43,45,51;
Volume using cross sections, Volume by disc for rotation about x-axis and y-axis, Examples 4,5,7,8,9,10;
Exercise: 6.1 1, 3, 19, 21, 23, 35, 37, 47, 49
Volume using cylindrical shells, Volume slicing by parallel planes, Examples 1,2,3; Arc length, Examples 1,2,3;
Exercise: 6.3 1,3,5,7
Area of surface revolution (revolution about x-axis and y-axis), Examples 1,2
Related problems.

Unit 6 Reduction Formula and Beta and Gamma Functions [15 Lecture Hrs]
Reduction Formula for, Beta and Gamma Functions and their properties, integral, all the solved examples and complete exercise [From the book "Integral Calculus" by "G.D. Pant and G.S. Shrestha", Sunila Prakashan]
Related problems.

Unit 7 Multiple Integrals [15 Lecture Hrs]
Double Integrals: Introduction, Double integral as a volume, Fubini's theorem for double integral, Examples 1,2,
Exercise -15.1: 1,3,5,7,9,11,15,17;
Iterated integrals, Double integral over general region, Fubini's theorem(stronger form), finding limit of integration, Example 1, Properties of double integral ,
Exercise-15.2: 1-5, 19,21,23,33,35,37,
Area by double integral, Examples 1,2,3; Double integral in polar form, Examples 1,2;
Exercise-15.4: 9,11,13,23,25
Triple integrals in Cartesian form, finding limit of integration, Examples 1,2,3
Exercise-15.5:7,9,11,13,15
Related problems.

Unit 8 Partial Derivatives [15 Lecture Hrs]
Function of several variables, Graphs, Level curves and Contours of functions of two variables, Limit for function of two variables, Properties of limit of function of two variables, Examples 1,2,3; Continuity, Example 5, Two path test for the existence of limit, Example 6,
Exercise-14.2: 1,3,4,5,13,25,31,32,33;
Partial derivatives, Examples 1,2,3,4,6; The mixed derivative theorem, Examples 10,11,
Exercise-14.3:1,3,5,12,16,23,25;
The chain rule, Examples 1,2,3,4,5,6;
Exercise-14.4: 1,3,25,27,33;
Directional Derivatives and gradient vector, Examples 1,2; Properties of directional derivatives, Example 3, Gradients and tangent to level curve, Example 4, Algebra rule for gradient, Examples 5,6 ;
Exercise-14.5: 1,3,7,11,12,13;
Tangent plane and differentials, Examples 1,2,3,4,5,6, Total derivatives and Euler theorem, Worked Out Examples 2,3,4,5,6,7, Exercise10(II)[From "Differential Calculus" by M.B. Singh and B.C. Bajracharya]
Related problems.

Unit 9 Extreme Value of a Function of two or More Variables [15 Lecture Hrs]
Criterion for a function to have extreme values, Derivative test for local extremum values, Saddle point, Examples 1,2,3,4,5; Absolute extrema on a closed bounded region, Examples 6,7,
Exercise- 14.7: 1 to 9, 31,33;
Lagrange's multiplier constraints maxima and minima, Examples 1,2,3,4;

Exercise-14.8: 1,2,3,6
Related problems.

Unit 10 Vector Calculus [15 Lecture Hrs]

Line integrals, Examples 1,2,3,
Exercise-16.1: 1,2,3;
Vector field, gradient field, Line integral of vector field, Examples 1,2,3,4,5; Flow integrals and circulation for velocity field, Examples 6,7; Flux across a simple closed plane curve, Example 8,
Exercise-16.2: 1,2,3,7,9,11,23;
Path independence, Line integrals in conservative field, Example 1, Greens theorem in the plane, Theorem-4 and 5 ,Examples 2,3,4,5,
Exercise-16.4:1,3,5,7;
Surface and area, Examples 1,2,4,5.
Related problems.

Text/ Reference Books:

1. M. D. Weir, J. Hass, F. R. Giordano: *Thomas Calculus*, Thirteenth Edition, Pearson.
2. J. Stewart; *Calculus Early Transcendental*, Seventh Edition, Cengage Learning India Private Limited.
3. G.D. Pant and G.S. Shrestha; *Integral Calculus and Differential Equations*, Sunita Prakashan, Kathmandu.
4. M.B. Singh and B.C. Bajracharya ; *A Text Book of Differential Calculus*, Sukunda Pustak Bhandar, Kathmandu.

Examination:

There will be a final examination of 70 marks for the period of three hours. The internal examination of 30 marks will be conducted by the department of mathematics of related campus and the marks will be submitted to Tribhuvan University Office of the Controller of Examination, Balkhu. A candidate must pass the internal and the final examinations separately.

Marks allocation for the internal examination:

- Written examinations: 20 marks (1 hour)
- A student or a group of students with presentation: 5 marks
- Assignments: 5 marks

Guidelines to the question setter:

In the final examination

1. Questions must include every unit.
2. There will be two groups, namely, Group A and Group B.
3. In group A, there must be OR selection for 15 marks questions.
4. In group B , there must be OR selection for 20 marks questions.
5. OR Selection must be given from the same unit.
6. Questions must be creative and should be appropriate to the allocated time.

On the basis of the guidelines mentioned, we enclose one set of model question for Claculus (Math 421)

Model Question

Tribhuvan University
Faculty of Humanities and Social Sciences

Bachelor Level/ I year/Humanities — Full Marks: 70
Mathematics (Math 421) — Pass Marks: 28
Calculus — Time: 3 hours

Candidates are required to give their answers in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt **All** the questions.

Group A $[2 \times 15 = 30]$

1. (a). Define asymptote to a curve. Find the horizontal and vertical asymptotes of $y = \dfrac{x+3}{x+2}$. [5]

   (b) Using Leibnitz theorem, if $y = \tan^{-1} x$, prove that $(1 + x^2)y_{n+1} + 2xy_n + n(n-1)y_{n-1} = 0$. [5]

   (c) Using Simpson's rule, evaluate:$\int_0^1 5x^4 dx; n = 4$. Also, find the upper bound for the error. [5]

   OR

   Define improper integral of first kind and second kind. Evaluate: $\int \dfrac{dx}{1+x}$. [5]

2. (a) Define Gamma function. Prove that $\Gamma(\dfrac{1}{2}) = \sqrt{\pi}$. [5]

   OR

   Obtain the reduction formula for $\int \tan^n x dx$ and hence find $\int \tan^5 x dx$. [5]

   (b) Verify Euler's theorem for the function $y = x^2 \tan^{-1}(\dfrac{y}{x})$. [5]

   OR

   Define partial derivatives. State its geometry. Find the values of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at the point (4,–5) if $f(x, y) = x^2 + 3xy + y–1$. [5]

   (c) Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$. [5]

Group B $[4 \times 10 = 40]$

3. (a) Sketch the graph of the function $y = x^4 - 4x^3 + 10$. [5]

   b) Using Maclaurin series, expand sinx in the series $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + ...$ to ∞ . [5]

4. (a) Find the area enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$. [5]

   (b) Find the arc length of the graph $f(x) = \dfrac{x^3}{12} + \dfrac{1}{x}; 1 \leq x < 4$. [5]

5. Find the volume of the region D enclosed by the surface $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$. [10]

   OR

   (a) Evaluate: $\int_0^1 \int_y^1 x^2 e^{xy} dxdy$.

   (b) Evaluate: $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dzdydx$. [4+6]

6. State Green's theorem in both forms. Verify both forms of Green's Theorem for the vector field $f(x, y) = (x - y)\overrightarrow{i} + x\overrightarrow{j}$ and the region R bounded by the unit circle $C : \overrightarrow{r} = \cos t\overrightarrow{i} + \sin t\overrightarrow{j}; 0 \leq j \leq 2\pi$. [2+8]

   OR

   State Stoke's theorem. Evaluate $\int\int \nabla \times F.nds$ for the hemisphere $S : x^2 + y^2 + z^2 = 9, z \geq 0$, its boundary circle $C : x^2 + y^2 = 9, z = 0$ and field $F = y\overrightarrow{i} + x\overrightarrow{j}$ and verify the Stoke's theorem. [2+8]

2 Math 422 Detailed Course of Analytic Geometry and Vectors

TRIBHUVAN UNIVERSITY
Faculty of Humanities and Social Sciences
Micro Syllabus

Course Title: Analytical Geometry and Vectors (Compulsory) **Full Marks:** 100
Course No.: Math 422 **Pass Mark:** 40
Level: B.A. **Year:** I
Nature of Course: Theory **Periods:** 9 Lectures Hrs/Week

Course Objectives: This course is designed for the first year of Four Years B.A. Program as a compulsory subject in mathematics. The objective of this course is to acquaint students with the basic concepts, and to build good knowledge base in Analytical Geometry of two and three dimensions and Vector Analysis.

Detailed Course Contents:

Unit 1 Transformation of Coordinates [10 Lecture Hrs]
1.1 Introduction of polar,
1.2 Cylindrical and spherical coordinates,
1.3 Change of origin of coordinates without change of the direction of axes,
1.4 Change of direction of axes without change of origin,
1.5 Change of direction of axes along with change of origin,
1.6 Invariants in orthogonal transformation,
1.7 All the solved examples,
1.8 All the problems of exercise 1 and related problems.

Unit 2 Ellipse [15 Lecture Hrs]
2.1 Introduction of conic section,
2.2 Derivation of equation in standard form,
2.3 Auxiliary circles and eccentric angles,
2.4 Equation of tangent and normal,
2.5 Condition of tangency,
2.6 Condition for a line to be a normal to the ellipse,
2.7 Director circle,
2.8 Some geometric properties of an ellipse,
2.9 Number of tangents from a point to an ellipse,
2.10 Chord of contact,
2.11 Pole and polar and their properties,
2.12 Conjugate diameters, Coordinates of the extremities of two conjugate diameters,
2.13 Sum of squares of conjugate semi diameters,
2.14 All the solved examples,
2.15 Exercise 2.1, 2.2, 2.3, 2.4 and related problems.

Unit 3 Hyperbola [15 Lecture Hrs]
3.1 Introduction,
3.2 Derivation of equation in standard form,
3.3 Position of a point,
3.4 Conjugate hyperbola,
3.5 Parametric representation,
3.6 Equation of tangent to the hyperbola,
3.7 Asymptotes to the hyperbola,
3.8 Condition of tangency,
3.9 Locus of the middle point s of the system of parallel chords having slope m,
3.10 Equation of rectangular hyperbola,
3.11 Equation of tangent to the rectangular hyperbola,
3.12 All the solved examples,
3.13 Exercise 3.1, 3.2 and related problems.

Unit 4 General equation of the second degree [15 Lecture Hrs]
4.1 General equation of second degree and conic represented by them,
4.2 Condition for general equation of second degree in x and y to represent conic,

4.3 Nature of conic,
4.4 Centre of conic, Equation of tangent to conic,
4.5 Equation of a normal to conic,
4.6 Condition of tangency,
4.7 Equation of pair of tangents,
4.8 Equation of asymptotes,
4.9 Director circle,
4.10 Pole and polar of a conic,
4.11 Conjugate diameters,
4.12 Intersection of conic,
4.13 All solved examples,
4.14 Exercise 5.1.5.2,5.3, 5.4 and related problems.

Unit 5 Straight Lines [20 Lecture Hrs]
5.1 Review of space and planes,
5.2 Representation of a line as a intersection of two plane,
5.3 Line in symmetric form,
5.4 Line through two points,
5.5 Reduction of the general form to the symmetrical form,
5.6 Perpendicular distance of a point from a line,
5.7 Condition for a line to lie in a plane,
5.8 General equation of a plane containing a line,
5.9 Coplanar lines and condition for it,
5.10 Skew lines,
5.11 Magnitude and equation of the line of the shortest distance between the two skew lines,
5.12 All the solved examples,
5.13 Exercise 3(a), 3(b),3(c), 3(d), 3(e) and related problems.

Unit 6 Sphere [15 Lecture Hrs]
6.1 Definition and equation of sphere,
6.2 Representation by the general equation of second degree,
6.3 Sphere through four given points,
6.4 Plane section of a sphere,
6.5 Intersection of two spheres,
6.6 Sphere with a given diameter,
6.7 Tangent plane and condition of tangency,
6.8 All the solved examples,
6.9 Exercise 4(a), 4(b), 4(c) and related problems.

Unit 7 Cone and Cylinder [15 Lecture Hrs]
7.1 Definition and equation of cone,
7.2 Condition that the general equation of second degree to represent a cone,
7.3 Condition that a cone has three mutually perpendicular generators,
7.4 Tangent lines and tangent plane,
7.5 Condition of tangency,
7.6 Reciprocal cone,
7.7 Enveloping and right circular cone,
7.8 Cylinder and enveloping cylinder,
7.9 Right circular cylinder,
7.10 All the solved examples,
7.11 Exercise 5(a), 5(b), 5(c), 5(d), 5(e) and related problems.

Unit 8 Product of three or more vectors [15 Lecture Hrs]
8.1 Multiplication of three vectors,
8.2 Scalar triple product,
8.3 Applications and geometrical meaning,
8.4 Properties of scalar triple product,
8.5 Condition of coplanarity of three vectors,
8.6 Vector triple product,
8.7 Scalar product of four vectors and vector product of four vector,
8.8 Reciprocal system of vector,
8.9 Properties of reciprocal system of vectors,

8.10 All the solved examples,
8.11 Exercise 2(A), 2(B), 2(C),2(D) and related problems.

Unit 9 Differentiation and integration of vectors [15 Lecture Hrs]
9.1 Vector function of a single variable,
9.2 Vector function and its expression in terms of unit vectors,
9.3 Limit and continuity of vector functions,
9.4 Differentiation of a vector function with respect to a scalar,
9.5 Differentiation of the product of a scalar and a vector,
9.6 Differentiation of a scalar product and vector product of two and three vectors,
9.7 Vector integrations,
9.8 All the solved examples,
9.9 Exercise 3(A), 3(B) and related problems.

Unit 10 Gradient, Curl and Divergence [15 Lecture Hrs]
10.1 Point functions,
10.2 Gradient of a scalar function,
10.3 Divergence of a vector function,
10.4 Curl of a vector function and their physical meaning and properties,
10.5 All the solved examples,
10.6 Exercise 4 and related problems.

Text/ Reference Books:

1. For unit 1, 2, 3, 4 : *Analytical Geometry* by M. R. Joshi, Jeevan Kafle (Sukunda Pustak Bhawan)
2. For unit 5, 6, 7: *Three Dimensional Geometry* by Y. R. Sthapit , B. C. Bajracharya (Sukunda Pustak Bhawan)
3. For unit 8, 9, 10: *Vector Analysis* by M. B. Singh , B. C. Bajracharya (Sukunda Pustak Bhawan)

Examination:
There will be a final examination of 70 marks for the period of three hours. The internal examination of 30 marks will be conducted by the department of mathematics of related campus and the marks will be submitted to Tribhuvan University Office of the Controller of Examination, Balkhu. A candidate must pass the internal and the final examinations separately.

Marks allocation for the internal examination:

- Written examinations: 20 marks (1 hour)
- A student or a group of students with presentation: 5 marks
- Assignments: 5 marks

Guidelines to the question setter:

In the final examination
1. Questions must include every unit.
2. There will be two groups, namely, Group A and Group B.
3. In group A, there must be OR selection for 15 marks questions.
4. In group B , there must be OR selection for 20 marks questions.
5. OR Selection must be given from the same unit.
6. Questions must be creative and should be appropriate to the allocated time.
7. In this paper the marks are divided as follows:
Two dimensional geometry- 25 marks
Three dimensional geometry-25 marks
Vectors - 20 marks

On the basis of the guidelines mentioned, we enclose one set of model question for Analytical Geometry and vectors (Math 422)

Model Question

Tribhuvan University
Faculty of Humanities and Social Sciences

Bachelor Level/ I year/Humanities — Full Marks: 70
Mathematics (Math 422) — Pass Marks: 28
Analytical Geometry and vectors — Time: 3 hours

Candidates are required to give their answers in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt **All** the questions.

Group A [$2 \times 15 = 30$]

1. (a) What does the equation $(x-a)^2 + (y-b)^2 = c^2$ become when it is transferred to parallel axes through the point $(a-c, b)$? [5]
   (b) Find the centre, eccentricity, foci, length of latus rectum and length of axes of the ellipse $x^2 + 4y^2 - 4x + 24y + 24 = 0$. [5]

   OR

   Find the equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point (x_1, y_1). [5]
   (c) Define conic section. Show that the equation $\frac{x^2}{9-c} + \frac{y^2}{5-c}$ represents a hyperbola if $5 < c < 9$. Also, find the coordinates of the foci of this hyperbola. [5]

2. (a) Obtain the equation of the cylinder which passes through $y^2 = 4ax, z = 0$ and whose generators are parallel to the line $x = y = z$. [5]

   OR

   Define reciprocal cone. Prove that the cones $ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$ are reciprocal.

   [5]

   (b) Prove that the necessary and sufficient condition for the vector function $\overrightarrow{a}$ of a scalar variable to have a constant direction is $\overrightarrow{a} \times \frac{d\overrightarrow{a}}{dt} = 0$. [5]

   OR

   The acceleration of a moving particle at any time t is given by $\frac{d^2\overrightarrow{r}}{dt^2} = 12\cos 2t\overrightarrow{i} - 8\sin 2t\overrightarrow{j} + 16t\overrightarrow{k}$. Find the velocity $\overrightarrow{v}$ and the displacement $\overrightarrow{r}$ at any time t if $t = 0, \quad v = 0, \quad r = 0$ [5]
   (c) Define divergence and curl of a vector function. If $f = x^3 + y^3 + z^3 - 3xyz$, find div(grad f) and curl(grad f). [5]

Group B [$4 \times 10 = 40$]

3. State the conditions under which general equation of second degree in x and y represent a parabola, an ellipse and a hyperbola. What conic does the equation $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$ represent? Find the centre of the conic. [3+3+4]

   OR

   Define pole and polar with respect to a conic. Find the equation of the polar with respect to the conic represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. Find the polar of the point $(2, 3)$ with respect to the conic $x^2 + 4xy - y^2 + 2x - 4y + 5 = 0$. [2+6+2]

4. Define skew lines and lines of shortest distance. Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$. Also, find the equation of the shortest distance.

[2+4+4]

OR

(a) Find the point where the line joining the points $(2, 1, 3)$ and $(4, -2, 5)$ cuts the plane $2x + y - z = 3$.

[5]

(b) Find the condition that the line $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ may lie in the plane $ax + by + cz + d = 0$.

[5]

5. Define great and small circle of plane section of a sphere. Find the equation of sphere having the circle $x^2 + y^2 + z^2 = 9$, $x - 2y + 2z = 5$ as a great circle. Also, find the centre and radius of the circle. [3+5+2]

6. (a) Define dot product of three vectors. If $\vec{a} = -\vec{i} + \vec{j} + \vec{k}$, $\vec{b} = \vec{i} - \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + \vec{j} - \vec{k}$, find $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$ and $\vec{a} \times (\vec{b} \times \vec{c})$. [5]

(b) Find the reciprocal system of the vectors $2\vec{i} + 3\vec{j} - \vec{k}$, $-\vec{i} + 2\vec{j} - 3\vec{k}$ and $3\vec{i} - 4\vec{j} + 2\vec{k}$. [5]

3 Math 423 Detailed Course of Linear Algebra

TRIBHUVAN UNIVERSITY
Faculty of Humanities and Social Sciences
Micro Syllabus

Course Title: Linear Algebra (Compulsory)	**Full Marks:** 100
Course No.: Math 423	**Pass Mark:** 40
Level: B.A.	**Year:** II
Nature of Course: Theory	**Periods:** 9 Lectures Hrs/Week

Course Objectives: This course is designed for the second year of Four years B.A. program as a compulsory subject in mathematics. The main objectives of this course are to enable the students to develop in-depth knowledge and good theoretical background in linear algebra, to take up higher studies to sustain interest and enjoyment of linear algebra and its applications in various branches of mathematics related to humanities and social sciences, and to get associated with teaching and familiar with recent trends in the field of Linear algebra.

Detailed Course Contents:

Unit 1. Matrix Operations [13 Lecture Hrs]

Matrix addition and scalar multiplication;
Definition, examples and conformal to addition and scalar multiplication,
Matrix- matrix multiplication;
Definition, example and conformal to multiplication,
Definition and example of Pre-multiplication and post- multiplication of matrix.
Theorem (No proof): The (i,j) -element in the product of two matrices $\mathbf{A}$ and $\mathbf{B}$ can be written in summation notation or the dot product of row i in $\mathbf{A}$ with column j in $\mathbf{B}$: $(\mathbf{AB})_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = \mathbf{r_i b_j}$,
Example, Special matrices; definition of diagonal matrix,
Kronecker delta,
Upper and lower triangular matrix, examples
Theorem: For any $n \times n$ matrix $\mathbf{A}$, we have $\mathbf{AI_n} = \mathbf{I_n A} = \mathbf{A}$.
Theorem (No proof): The set of upper triangular matrices in $\mathbb{R}^{n\times n}$ is a subspace of $\mathbb{R}^{n\times n}$. That is, it is a vector space in its own right. Example
Theorem: The product of two upper triangular matrices (of the same size) is also upper triangular.
Matrix transpose; definition and example
Theorem: If the product $\mathbf{AB}$ of two matrices $\mathbf{A}$ and $\mathbf{B}$ exists, then $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B^T A^T}$.
Theorem:
(a) $(\mathbf{A^T})^{\mathbf{T}} = \mathbf{A}$.
(b) $(\mathbf{A}+\mathbf{B})^{\mathbf{T}} = \mathbf{A^T} + \mathbf{B^T}$
(c) $(c\mathbf{A})^{\mathbf{T}} = c\mathbf{A^T}$.
Symmetric matrices; definition and example
Skew- symmetric matrices; definition and examples, relation between them, Non- commutativity of matrix multiplication; justifying $\mathbf{AB} \neq \mathbf{BA}$ by example and Associativity law for matrix multiplication
Theorem: Matrix multiplication is associative. Thus, if the product exists, we have
$\mathbf{(AB)C = A(BC)}$ and $\mathbf{A(B+C) = AB+AC}$.
Trace of a square matrix and its properties
Elementary matrices;
Theorem (No proof): If the matrix $\widetilde{A}$ is the result of applying a row operation R to the matrix $\mathbf{A}$, and if $\mathbf{E}$ is the matrix that results from applying R to $\mathbf{I}$, then $\widetilde{A} = \mathbf{EA}$,
Vector- matrix product;discussion and example
Theorem (No proof): If $\mathbf{y^T}$ is a row vector, then $\mathbf{y^T A}$ is a linear combination of the rows of $\mathbf{A}$, with coefficients taken to the components of the row vector $\mathbf{y^T}$: $\mathbf{y^T A} = [y_1, y_2, ..., y_n]\ \mathbf{A} = y_1\mathbf{r_1} + y_2\mathbf{r_2} + ... + y_n\mathbf{r_n}$ where $\mathbf{r_i}$ are the rows of $\mathbf{A}$.

Theorem: If the rows of $\mathbf{A}$ are denoted by $\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_m}$, then $\mathbf{AB} = \begin{bmatrix} \mathbf{r_1} \\ \mathbf{r_2} \\ . \\ . \\ . \\ \mathbf{r_m} \end{bmatrix} \mathbf{B} = \begin{bmatrix} \mathbf{r_1 B} \\ \mathbf{r_2 B} \\ . \\ . \\ . \\ \mathbf{r_m B} \end{bmatrix}$.

Solving systems with a left inverse;
Solving systems with a right inverse.
Theorem (No proof): Suppose that two matrices $\mathbf{A}$ and $\mathbf{B}$ have the property that $\mathbf{AB} = \mathbf{I}$. Then the

system of linear equations $\mathbf{Ax} = \mathbf{b}$ has at least one solution, namely, $\mathbf{x=Bb}$
Theorem (No proof): If the matrix **A** has a right inverse, then for each **b** the equation $\mathbf{Ax= b}$ has at least one solution. If **A** has a left inverse, then that equation has at most one solution.
Theorem: If **A** and **B** are square matrices such that $\mathbf{BA= I}$, then $\mathbf{AB= I}$.
Theorem: If **A**, **B**, and **C** are square matrices such that $\mathbf{AB} = \mathbf{AC= I}$, then $\mathbf{B= C}$
Invertible matrices; definition of invertible matrix, singular and non-singular matrices.
Theorem: If **A**, **B** and **C** such that $\mathbf{AC= BA= I}$, then $\mathbf{B= C}$
Examples
Theorem: For invertible $n \times n$ matrices **A** and **B**, we have $(\mathbf{AB})^{-1} = \mathbf{B^{-1}A^{-1}}$.Thus, the product of two invertible matrices is invertible. The result extends to the product of any number of invertible matrices.
Similar matrices; definition and examples
Equivalence relations; discussion and examples,
Rank of a Matrix.
Related problems.

Unit 2. Determinants [13 Lecture Hrs]

Properties of determinant; definition of determinant, notation, difference between matrix and determinant,
Linear function, properties, examples
Theorem: The function Det : $M_{2\times2}(F) \to F$ is a linear function of each row of a 2×2 matrix when the other row is held fixed. That is, if u, v, and w are in F^2 and k is a scalar, then det$\begin{pmatrix} u+kv \\ w \end{pmatrix}$= det$\begin{pmatrix} u \\ w \end{pmatrix}$+k det$\begin{pmatrix} v \\ w \end{pmatrix}$ and det$\begin{pmatrix} w \\ u+kv \end{pmatrix}$= det$\begin{pmatrix} w \\ u \end{pmatrix}$+k det$\begin{pmatrix} w \\ v \end{pmatrix}$
Theorem: Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2\times2}(F)$. Then the determinant of A is nonzero if and only A is invertible. Moreover, if A is invertible, then $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$
corollary: A matrix $A \in M_{2\times2}(F)$ is invertible if and only if $\det(A) \neq 0$. Furthermore, if A is invertible, then $\det(A^{-1} = \frac{1}{\det(A)}$
Theorem: If A has two identical rows(columns), then $|A| = 0$
Theorem: Let $A \in M_{n\times n}(F)$, and let B be a matrix obtained by adding a multiple of one row of A to another row of A.Then $|B| = |A|$
Theorem: If $A \in M_{n\times n}(F)$ and B is a matrix obtained from A by interchanging any two rows, then $|B| = -|A|$
In other words, if the matrix $\widetilde{A}$ is obtained from the matrix A by swapping a pair of rows, then Det($\widetilde{A}$)= - Det(A).
Example
Theorem (No Proof): There is a one and only one determinant function having properties I, II, and III.
Examples
Theorem : The determinant of a 2×2 matrix follows this rule:
Det $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.
Algorithm without scaling: examples
Zero determinant; discussion
Theorem (No proof): The determinant of a matrix is 0 if and only if the rows (or columns) of the matrix form a linearly dependent set.
Examples
Theorem: A square matrix is invertible (non-singular) if and only if its determinant is nonzero.
Examples
Calculating areas and volumes; discussion
Theorem: In $\mathbb{R}^2$, if a triangle has vertices **0**, **u**, and **v**, then area of the triangle is one-half of the absolute value of the 2×2 determinant having rows (or columns) **u** and **v**:
Area $[\Delta(\mathbf{0}, \mathbf{u}, \mathbf{v})] = \frac{1}{2} \left| \text{Det} \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right|$.
Examples
Theorem: The area of a parallelogram generated by the three points **0**, **u**, and **v** in $\mathbb{R}^2$ is the absolute value of the determinant having rows (or columns) **u** and **v**:
Area (**0**, **u**, **v**, **u+v**)= $\left| \text{Det} \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right|$.
Theorem: Let S be a triangle or parallelogram in $\mathbb{R}^2$ having vertex at **0**. Let T be a linear mapping from

$\mathbb{R}^2$ to $\mathbb{R}^3$, given by matrix $\mathbf{A}$, so that T($\mathbf{x}$)= $\mathbf{Ax}$. Then Area $[T(S)] = |\text{Det}(\mathbf{A})|$. Area (S).
Example and Minors and cofactors; definition and examples
Theorem (No proof): Let $\mathbf{A}$ be an $n \times n$ matrix whose minor are denoted by $\mathbf{M}^i$. Then these formulas are valid:
Det($\mathbf{A}$)= $\sum_{j=1}^{n}(-1)^{i+j}a_{ij}Det(\mathbf{M^{ij}})$ (expansion using row i)
Det($\mathbf{A}$)= $\sum_{i=1}^{n}(-1)^{i+j}a_{ij}\text{Det}(\mathbf{M^{ij}})$ (expansion using column j)
Examples and Definition of cafactor expansion
Theorem (No proof): Let $\mathbf{A}$ be an $n \times n$ matrix whose cofactors are denoted by C_{ij}. Then these formulas are valid:
Det($\mathbf{A}$)= $\sum_{j=1}^{n} a_{ij}C_{ij}$) (expansion using row i)
Det($\mathbf{A}$)= $\sum_{i=1}^{n} a_{ij}C_{ij}$) (expansion using column j).
Examples and Product of determinants, Direct method for computing determinants.
Lemma: If $\mathbf{E}$ is an $n \times n$ elementary matrix and if $\mathbf{A}$ is any $n \times n$, then
Det($\mathbf{EA}$)= Det($\mathbf{E}$)Det($\mathbf{A}$)
Theorem: For square matrices A and B of the same size, we have
Det($\mathbf{AB}$)= Det($\mathbf{A}$) Det($\mathbf{B}$).
Examples
Theorem (No proof): Recall that the minors of the matrix $\mathbf{A}$ are denoted by $\mathbf{M^{ij}}(\mathbf{A})$. Then we have $\mathbf{M^{ij}}(\mathbf{A^T})= \left[\mathbf{M^{ij}}(\mathbf{A})\right]^T$.
Examples
Theorem: The determinant of a matrix and the determinant of its transpose are equal:
Det($\mathbf{A}$) = Det($\mathbf{A^T}$).
Theorem: The determinant of a matrix is a linear function of any one column (or row).
Examples and Cramer's rule
Theorem: Cramer's rule.
Examples
Computing inverses using determinants
Theorem (No proof) : The inverse of an $n \times n$ invertible matrix $\mathbf{A}$ can be computed by the the formulas $(\mathbf{A^{-1}})_{\mathbf{ij}} = \frac{(-1)^{i+j}}{\text{Det}(\mathbf{A})}\text{Det}(\mathbf{M^{ij}})= \frac{C_{ji}}{\text{Det}(\mathbf{A})}$.
Examples
Related problems.

Unit 3. System of linear Equations [13 Lecture Hrs]

Linear equation; definition, examples
System of linear equation; consistent, inconsistent, unique solution, no solution, infinitely many solutions and their examples
General solution of a system of equations; discussion and examples
Coefficient matrix, the vector of unknowns, the right hand-side vector, and the augmented matrix
Gaussian elimination; definition and examples
Elementary replacement and scale operations; definition and example
Row equivalent pairs of matrice, pivot row, row operation, row equivalent, equivalent matrices, and their examples, equivalence relation
Elementary row operations; definition, scaling of an equation, replacement, scale, swap
Redce row echelon form; definition, pivot,and their examples
Theorem (No proof): If the system of equation $\mathbf{Ax}= \mathbf{b}$ is consistent, and if a matrix $\mathbf{B}$ exists such that $\mathbf{BA}$=$\mathbf{I}$, then the system of equations has a unique solution, namely $\mathbf{x} = \mathbf{Bb}$.
Examples
Theorem (No proof): If $\mathbf{A}$ is invertible, then for any prescribed $\mathbf{b}$ the equation $\mathbf{Ax= b}$ has one and only one solution, namely $\mathbf{x} = \mathbf{A^{-1}b}$.
Theorem: Every matrix has one and only one reduced row echelon form
Pivot position and examples
Row echelon form; definition, difference between reduced and row echelon form with intuitive interpretation.
Row equivalent, examples
Consistent and inconsistent systems;
Theorem: A system of linear equations, $\mathbf{Ax}= \mathbf{b}$ is consistent if and only if the vector $\mathbf{b}$ is in the span of the set of columns of $\mathbf{A}$.
Theorem (No proof): Let A be an $m \times n$ matrix. The system of equations $\mathbf{Ax}= \mathbf{b}$ is consistent for all $\mathbf{b}$ in $\mathbb{R}^m$ if and only if the columns of $\mathbf{A}$ span $\mathbb{R}^m$. In other words, Col ($\mathbf{A}$)= $\mathbb{R}^m$
Theorem (No proof): Let two linear systems of equations be represented by their augmented matrices. If these two augmented matrices are row equivalent to each other, then the solutions of the two systems are

identical

Theorem: Let A be an $m \times n$ matrix. The system of equations $\mathbf{Ax}= \mathbf{b}$ is consistent for all $\mathbf{b}$ in $\mathbb{R}^m$ if and only if each row of the coefficient matrix $\mathbf{A}$ has a pivot position.

Theorem: A system of linear equations is consistent if and only if its augmented matrix has a pivot position in the last column.

Theorem (No proof): A system of linear equations is consistent if and only if the reduced row echelon form of its augmented matrix does not have a pivot position in the last column.

Related problems.

Unit 4. Vectors and Matrices [13 Lecture Hrs]

n-Tuples and vectors; definition and examples

Vector addition and multiplication by scalars; definition and examples

Linear combinations; definition and examples

Matrix- matrix product ; definition and examples

Indexed sets of vectors:

Linearly dependence and independence; definition, examples.

Theorem: If an indexed set of two or more vectors is linearly dependent, then some vectors in the set is a linear combination of the others.

Theorem: If an indexed set of two or more vectors in $\mathbb{R}^m$ is linearly dependent, then some vector in the list is a nontrivial linear combination of vectors preceding it in the list.

Theorem: The rows of a matrix form a linearly dependent set if and only if there is a zero row in any row in any row echelon form of that matrix,

Using the row reduction process; example

Theorem (No proof): The column vectors of a matrix form a linearly dependent set if and only if there is a column having no pivot.

Theorem (No proof): The column vectors of a matrix form a linearly independent set if and only if there is a pivot position in each column of the matrix.

Examples

Determining linear dependence or independence.

Kernel or null spaces of a matrix; definition and examples

Theorem: If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbf{Ker(A)}$, and if α is a scalar, then $\mathbf{x}+\mathbf{y}$ and $\alpha\mathbf{x}$ are also in $\mathbf{Ker(A)}$.

Theorem: If $\mathbf{Ax}= \mathbf{b}$ and $\mathbf{y} \in \mathrm{Ker}(\mathbf{A})$, then $\mathbf{A(x+y) = b}$.

Theorem: If $\mathbf{Ax}= \mathbf{Ay}$, then $\mathbf{x} - \mathbf{y} \in \mathrm{Ker}(\mathbf{A})$.

Theorem (No proof): If $\mathbf{u}$ is a vector such that $\mathbf{Au}= \mathbf{b}$, then every solution of the equation $\mathbf{Ax}= \mathbf{b}$ is of the form $\mathbf{x} = \mathbf{u} + \mathbf{z}$, for some vector $\mathbf{z}$ in $\mathbf{Ker(A)}$.

Theorem: If two matrices are row equivalent to each other, then their kernels are equal.

Homogeneous equations; definition and examples

Uniqueness of the reduced row echelon form.

Theorem: Every matrix has one and only one reduced row echelon form

Rank of a matrix; definition and examples

Corollary (No proof): The rank of a matrix is the number of pivots in its reduced row echelon form, which is the same as the number of pivot positions in the matrix.

examples.

Theorem: The rank of an $m \times n$ matrix cannot be greater than n or m. In symbols, we have $\mathrm{Rank}(\mathbf{A}) \leq \min(m, n)$.

Matrix- vector products; definition and examples.

Theorem (No proof): The following properties of an $m \times n$ matrix $\mathbf{A}$ are equivalent to each other.

(a) The rank of $\mathbf{A}$ is less than n,

(b) The reduced row echelon form of $\mathbf{A}$ has fewer than n nonzero rows,

(c) The matrix $\mathbf{A}$ has fewer than n pivot positions,

(d) At least one column in $\mathbf{A}$ has no pivot positions,

(e) There is at least one free variable in the system of equations $\mathbf{Ax}= \mathbf{0}$,

(f) The system $\mathbf{Ax}= \mathbf{0}$ has some nontrivial solutions.

Corollary (No proof): A homogeneous system of linear equations in which there are more variables than equations must have some nontrivial solutions

General solution of a system; example

Theorem: Each vector in the span of a linearly independent set (in a vector space) has a unique representation as a linear combination of elements of that set.

examples

Theorem (No proof): Any (indexed) set of more than n vectors in $\mathbb{R}^n$ is necessarily linearly dependent.

Theorem (No proof): Let $\{\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_m}\}$ be a linearly dependent indexed set of at least two vectors (in

a vector space). Then some vectors in the indexed list is a linear combination of preceding vectors in that list.
Related problems.

Unit 5. Vector Spaces and Subspaces [18 Lecture Hrs]

Introduction; vectors; definition and examples
Linear combinations of vectors; definition, examples,
Basic properties of the vector spaces $\mathbb{R}^n$
Properties of $\mathbb{R}^n$ as a vector space;
Span of a set of vectors; definition and examples
Geometric interpretation of vectors
Line passing through origin
Lines in $\mathbb{R}^2$; parametric line, and examples
Lines in $\mathbb{R}^3$; examples
Theorem: Let $\mathbf{L_1}$ and $\mathbf{L_2}$ be two lines in $\mathbb{R}^n$ describe as
$\mathbf{L_1} = \{\mathbf{u} + t\mathbf{v} : t \in \mathbb{R}\}$
$\mathbf{L_2} = \{\mathbf{w} + s\mathbf{z} : s \in \mathbb{R}\}$. These lines are the same if and only if **u-w** and **v** are multiples of **z**, examples
Lines and planes in $\mathbb{R}^n$; examples
The span of a set of vectors; definition, examples
Theorem: If a vector space is spanned by some set of n vectors, then every set of more than n vectors in that space must be linearly dependent.
Theorem: Let S be a linearly independent set in a vector space V. If $\mathbf{x} \in V$ and $\mathbf{x} \notin \text{Span}(S)$, then $S \cup \{x\}$ is linearly independent.
Example
Theorem: In any vector space, if c is a scalar, then c$\mathbf{0} = \mathbf{0}$.
Theorem: In a vector space, if $\mathbf{x}$ is a vector and c is a scalar such that c$\mathbf{x}= \mathbf{0}$, then either c $= 0$ or $\mathbf{x} = \mathbf{0}$.
Theorem: In a vector space, for each $\mathbf{x}$, the point $\widetilde{\mathbf{x}}$ is uniquely determined where $\widetilde{\mathbf{x}}$ is the inverse of $\mathbf{x}$.
Theorem: In a vector space, every vector $\mathbf{x}$ satisfies the equation $0\mathbf{x} = \mathbf{0}$.
Theorem: In any vector space, we have $(-1)\mathbf{x} = \widetilde{\mathbf{x}}$. where $\widetilde{\mathbf{x}}$ is the inverse of $\mathbf{x}$
Examples
Theorem: Let U be a subset of a vector space V. Suppose that U contains the zero element and is closed under addition and multiplication by scalars. Then U is a subspace of V.
Theorem: The span of a nonempty set in a vector space is a subspace of that vector space.
Examples
Theorem: The vector sum of two subspaces in a vector space is also a subspace.
Example
Theorem: The intersection of two subspaces in a vector space is also a subspace of that vector space.
But for union may not hold, counter Example
Related problems.

Unit 6. Linear Transformation [18 Lecture Hrs]

Functions, mappings, and transformations with definitions and examples;
Domain, co-domain, and range; definition and various examples
Injective and surjective mappings; definition and examples, bijective mapping, and examples,
Linear transformations; definition and examples
Theorem: Let $\mathbf{A}$ be an $m \times n$ matrix. The mapping $\mathbf{x} \mapsto \mathbf{Ax}$ is linear from $\mathbb{R}^n$ to $\mathbb{R}^m$.
Theorem: Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then there is an $m \times n$ matrix $\mathbf{A}$ such that T($\mathbf{x}$) = $\mathbf{Ax}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.
Theorem: A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is completely determined by the images of the standard unit vectors in $\mathbb{R}^n$.
Examples
Theorem: Let S and T be two linear transformations defined by S($\mathbf{y}$) = $\mathbf{Ay}$ and T($\mathbf{x}$) = $\mathbf{Bx}$. ($\mathbf{A}$ and $\mathbf{B}$ are matrices). If the composition SoT exists, then its matrix is $\mathbf{AB}$.
Injective and surjective linear transformations;
Theorem: Let $\mathbf{A}$ be an $m \times n$ matrix. For the linear map $\mathbf{x} \mapsto \mathbf{Ax}$ to be surjective (onto), it is necessary and sufficient that the columns of $\mathbf{A}$ span $\mathbb{R}^m$
Definition of kernel and its example
Theorem: In order that the linear map $\mathbf{x} \mapsto \mathbf{Ax}$ to be injective (one- to- one), it is necessary and sufficient that the kernel of $\mathbf{A}$ contain only the zero vector.
Effects of linear transformations; definition and examples, rotation, orthogonal projection, shear, reflection

Effects of transformations on geometrical figures;
Theorem: A linear transformation maps one line segment into another.
Theorem: In $\mathbb{R}^2$, a counterclockwise rotation of every point by an angle ϕ is a linear transformation whose matrix is
$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.
Theorem: If f is a linear transformation from a vector space X to a vector space Y, then for any subspace U in X, $f[U]$ is a subspace of Y.
Example
Definition of inverse image
Theorem: If f is a linear transformation from a vector space X to a vector space Y, then for any subspace V in Y, $f^{-1}[V]$ is a subspace in X.
Examples
Composition of two linear mappings; Definitions and examples
Theorem: Composition of two linear maps is also linear.
Theorem: Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{B}$ be an $n \times k$ matrix. Define S($\mathbf{x}$)= $\mathbf{Ax}$ and T($\mathbf{y}$)= $\mathbf{By}$. Then (SoT)($\mathbf{x}$)= $(\mathbf{AB})\mathbf{x}$.
Theorem: The kernel of a linear transformation is subspace in the domain of that transformation.
Theorem: The range of a linear transformation is a subspace of its co-domain.
Example
kernel(Null space) and image of linear transformation; definition and examples
The row space and column space of a matrix; definition of row space and column space.
Theorem (No proof): If two matrices are row equivalent to each other, then they have the same row space.
Examples
Theorem: The column space of an $m \times n$ matrix $\mathbf{A}$ is the set $\{\mathbf{Ax} : \mathbf{x} \in \mathbb{R}^{\mathbf{n}}\}$
Examples
Theorem (No proof): The row space of a matrix $\mathbf{A}$ is the column space of $\mathbf{A}^T$.
Examples

Basis for a vector space: definition and examples
Theorem: The set of columns of any $n \times n$ invertible matrix is a basis for $\mathbb{R}^n$
Theorem (No proof): If a basis is given for a vector space, then each vector in the space has a unique expression as a linear combination of elements in that basis.
Coordinate vector: definition and examples
Theorem: If β is a basis, then the mapping of $\mathbf{x}$ to its coordinate vector $[\mathbf{x}]_\beta$ is linear.
Isomorphism and equivalence relation; definition of equivalence relation
Theorem: The relation of isomorphism between two vector spaces is an equivalence relation.
Examples
Theorem (No proof): If a vector space has a finite basis, then all of its bases have the same number of elements.
Finite dimensional and infinite dimensional vector spaces; definitions and examples
Theorem (No proof): Two finite dimensional vector spaces are isomorphic to each other if and only if their dimensions are the same.
Examples
Theorem (No proof): The span of a set is not affected by removing from the set one element that is a linear combination of other elements of that set.
Theorem (No proof): If a set of n vectors spans an n- dimensional vector space, then the set is a basis for that vector space.
Theorem (No proof): In an n-dimensional vector space, every linearly independent set of n vectors is a basis.
Examples
Theorem (No proof): If a set of n vectors spans a vector space, then the dimension of that space is at most n.
Theorem (No proof): If a set of n vectors is linearly independent in a vector space, then the dimension of the space is at least n.
Theorem: The pivot column of a matrix form a basis for its column space.
Examples
Theorem (No proof): In a finite dimensional vector space, any linearly independent set can be expanded to create a basis.
Dimensions of various subspaces; definition of domain, co-domain, nullity and range and example.

Theorem: If T is a linear transformation whose domain is an n-dimensional vector space, then Dim(Ker(T))+ Dim(Range(T))= n.
Theorem (No proof): The row space and column space of a matrix have the same dimension: Dim(Row($\mathbf{A}$))= Dim(Column($\mathbf{A}$)).
Examples
Coordinate vectors; definition and example
Changing coordinates; examples
Theorem: If $\sum_{i=1}^{n} a_i\mathbf{u_i}= \mathbf{0}$ and if L is linear, then $\sum_{i=1}^{n} a_iL(\mathbf{u_i})= \mathbf{0}$
Examples
Mapping a vector space into itself.
Theorem: Let $\{\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_n}\}$ be a linearly independent set in some vector space. Let $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ be arbitrary vectors in another vector vector space. Then there is a linear transformation T such that T($\mathbf{u_i}$)= $\mathbf{v_i}$ for $1 \leq i \leq n$.
Theorem: If T is a linear transformation, then any linear dependence of the type $\sum_{i=1}^{n} a_i\mathbf{u_i} = \mathbf{0}$ must imply $\sum_{i=1}^{n} a_iT(\mathbf{u_i}) = \mathbf{0}$.
Related problems.

Unit 7. Eigen Systems [13 Lecture Hrs]

Introduction
Eigenvectors and eigenvalues; definition and example
Using determinants in finding eigenvalues;
Theorem: A scalar λ is an eigenvalue of a matrix $\mathbf{A}$ if and only if Det($\mathbf{A} - \lambda\mathbf{I}$) = 0
Examples
Linear transformations; definition and examples, definition of diagonalization and example.
Theorem (No proof): The geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity.
Distinct eigenvalues
Theorem: Any set of eigenvector corresponding to distinct eigenvalues of a matrix is linearly independent.
Corollary (No proof): If an $n \times n$ matrix $\mathbf{A}$ has n distinct eigenvalues, then $\mathbf{A}$ is diagonalizable.
Bases of eigenvectors.
Corollary (No proof): If an $n \times n$ real matrix $\mathbf{A}$ has n distinct real eigenvalues, then the corresponding eigenvectors of $\mathbf{A}$ constitute a basis for $\mathbb{R}^n$.
Characteristic equation and characteristic polynomial; definition and examples
Diagonalization involving complex numbers; examples
Theorem (No proof) : If a real matrix and has a complex eigenvalue λ , then the conjugate ($\overline{\lambda}$) is also an eigenvalue. Thus we have $\mathbf{Ax} = \lambda\mathbf{x}$ and $\mathbf{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$.
Application: powers of a matrix; examples
Related problems.

Unit 8. Inner Product Vector Spaces [13 Lecture Hrs]

Definition and properties of inner product spaces with examples. The norm in an inner product space: definition and examples
Theorem: In any inner-product space, the norm has these properties:
(a) $||\mathbf{x}|| > 0$ for every nonzero vector $\mathbf{x}$
(b) $||\alpha\mathbf{x}|| = |\alpha|||\mathbf{x}||$
(c) $| < \mathbf{x}, \mathbf{y} > | \leq ||\mathbf{x}||.||\mathbf{y}||$ (**Cauchy- Schwarz Inequality**)
(d) $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$ (**Triangle Inequality**)
(e) If $< \mathbf{x}, \mathbf{y} >= 0$, then $||\mathbf{x} + \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$ (**Pythagorean Theorem**)
(f) $||\mathbf{x} + \mathbf{y}||^2 = ||\mathbf{x}||^2 + < \mathbf{x}, \mathbf{y} > + < \mathbf{y}, \mathbf{x} > + ||\mathbf{y}||^2$,
$||\mathbf{x} + \mathbf{y}||^2 = ||\mathbf{x}||^2 + 2 < \mathbf{x}, \mathbf{y} > + ||\mathbf{y}||^2$, *in the real case*
Distance function: definition and examples
Mutually orthogonal vectors: definition and examples
Theorem: Every orthogonal set of nonzero vectors in an inner-product space is linearly independent.
Theorem (No proof): In $\mathbb{R}^n$, there does not exist an orthogonal set of $n + 1$ nonzero vectors.
Theorem (No proof): Any orthogonal set of n nonzero vectors in $\mathbb{R}^n$ is a basis $\mathbb{R}^n$.
Examples
Orthogonal projection.
Theorem: In an inner-product space, the orthogonal projection of a vector $\mathbf{x}$ onto a nonzero vector $\mathbf{y}$ is the point

$$\mathbf{p} = \frac{<\mathbf{x},\mathbf{y}>}{<\mathbf{y},\mathbf{y}>}\mathbf{y}.$$

It has the property that **x-p** is orthogonal to **y**. Thus, **x** is split into an orthogonal pair of vectors in the equation **x=p+ (x-p)**.
Angle between vectors: definition and examples
Orthogonal complements: definition and examples
Theorem: In an inner-product space, the orthogonal complement of a set is the same as the orthogonal complement of its span.
Examples
Theorem: In an inner-product space, the orthogonal complement of any subset is a subspace
Examples
Theorem: The null space of a matrix is the orthogonal complement of its row space:
$\text{Null}(\mathbf{A}) = [\text{Row}(\mathbf{A})]^{\perp}$ Examples
Orthogonal bases; discussion
Theorem: If $\{\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_n}\}$ is an orthonormal set in an inner-product space, then each **x** in the span of that set is expressible as
$\mathbf{x} = \sum_{i=1}^{n} < \mathbf{x}, \mathbf{u_i} > \mathbf{u_i}$.
Definition of orthogonal, unitary with examples.
Theorem (No proof): Every orthonormal set is linearly independent.
Theorem: $\{\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_n}\}$ be an orthonormal basis for a subspace U in an inner-product space. The orthogonal projection of any **x** onto U is the point
$\mathbf{p} = \sum_{i=1}^{n} < \mathbf{x}, \mathbf{u_i} > \mathbf{u_i}$
Theorem (No proof): In order that a vector be orthogonal to a subspace (in an inner-product space), it is sufficient that the vector be orthogonal to each member of a set that spans the subspace.
Subspaces in inner-product spaces; definition of direct sum and examples
Theorem: If U is a subspace in an n-dimensional inner-product space, then
$\text{Dim}(U) + \text{Dim}(U)^{\perp} = n$
Theorem: Let U and V be subsets of an inner-product space. If $U \subseteq V$, then $V^{\perp} \subseteq U^{\perp}$.
Theorem: If U is a finite-dimensional subspace in an inner- product space X, then X is the direct sum of U and $U^{\perp}$:
X= U + $U^{\perp}$,
$U \bigcap U^{\perp} = \{\mathbf{0}\}$,
$U = U^{\perp\perp}$
The Gram-Schmidt Orthogonalization Process with examples
Theorem: Every finite-dimensional inner-product space has an orthonormal basis.
Examples and Linear least-square solution.
Theorem (No proof): If a system of equation **Ax= b** is inconsistent, then the system of normal equation,
$\mathbf{A^T A x} = \mathbf{A^T b}$
can be solved for **x**, and with this choice of **x**, the point **Ax** will be as close to **b** as possible.
Examples
Related problems.

Unit 9. Matrix Factorizations [18 Lecture Hrs]

Hermitian matrix and self-adjoint mapping: definitions and examples.
Theorem: Every self-adjoint mapping is linear.
Theorem: All eigenvalues of a self-adjoint operator are real.
Theorem: Any two eigenvectors of a self-adjoint operator if associated with different eigenvalues, are mutually orthogonal.
Theorem: Consider $\mathbb{C}^n$ with its standard inner product, and let **A** be an $n \times n$ matrix. Define a linear transformation L by writing L(**x**)= **Ax**,where $\mathbf{x} \in \mathbb{C}^n$. The operator L is self-adjoint if and only if **A** is Hermitian.
Corollary (No proof): All eigenvalues of a Hermitian matrix are real.
Corollary (No proof): All eigenvalues of a symmetric real matrix are real.
Theorem: Let L be a linear transformation on a finite-dimensional vector space. If **A** is the matrix for L relative to a given basis, then the eigenvalues of L and **A** are the same.
Examples, Unitary and orthogonal matrices: definition and examples
Theorem (Cayley- Hamilton): Every square matrix satisfies its own characteristic equation.
Examples upto 3×3 matrix. Quadratic forms: definition and example
Theorem: If **A** and **B** are $n \times n$ real matrices connected by the relation
$\mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A^T})$
then the corresponding quadratic forms of **A** and **B** are identical, and **B** is symmetric.
Examples
Permutation matrix: definition and examples

LU-factorization: definition and examples
QR-factorization with examples
Theorem (No proof): The vector $\mathbf{x}$ is the least-squares solution of the system of equations $\mathbf{Ax}= \mathbf{b}$.
Theorem (No proof): Any $m \times n$ matrix of rank n has a QR- factorization, where $\mathbf{Q}$ is an $m \times n$ matrix, $\mathbf{Q^T Q} = \mathbf{I_n}$, and $\mathbf{R}$ is an $n \times n$, upper triangular, and invertible matrix.
Examples,
Partitioned matrices; definition and examples
Solving a system having a 2×2 block matrix.
Richardson iterative method and examples
Jocobi iterative method and examples, Gauss-Seidel method and example
Related problems.

Unit 10. Applications of Linear Algebra [13 Lecture Hrs]
Algorithm for the reduced row echelon form with examples,
Least square approximation with examples,
Traffic flow with examples,
Graph theory with examples,
Cholesky decomposition with examples,
Leontief economic model with examples,
Linear ordinary differential equations with examples,
Polynomial interpretation,
Zero of polynomial,
Rolle's theorem, Descartes rule of sign,
Newton's method of approximation with examples.
Related problems.

Text/Reference Books:

1. W. Cheney and D. Kincaid: *Linear Algebra: Theory and Applications*, Jones and Bartlett India Pvt. Ltd.
2. S.H. Friedberg, A.J. Insel and L.E. Spence: *Linear Algebra*, PHI Learning Pvt. Ltd.. New Delhi.
3. T.P. Nepal, T.B. Budhathoki and J. Kafle: *A Textbook of Linear Algebra*, Heritage Publication and Distributors Pvt. Ltd., Kathmandu.
4. R. M. Shrestha and S. Bajracharya: *Linear Algebra, Groups, Rings and Theory of Equations*, Sukunda Pustak Bhawan, Kathmandu.

Examination:
There will be a final examination of 70 marks for the period of three hours. The internal examination of 30 marks will be conducted by the department of mathematics of related campus and the marks will be submitted to Tribhuvan University Office of the Controller of Examination, Balkhu. A candidate must pass the internal and the final examinations separately.

Marks allocation for the internal examination:

- Written examinations: 20 marks (1 hour)
- A student or a group of students with presentation: 5 marks
- Assignments: 5 marks

Guidelines to the question setter:

In the final examination
1. Questions must include every unit.
2. There will be two groups, namely, Group A and Group B.
3. In group A, there must be OR selection for 15 marks questions.
4. In group B , there must be OR selection for 20 marks questions.
5. OR Selection must be given from the same unit.
6. Questions must be creative and should be appropriate to the allocated time.

On the basis of the guidelines mentioned, we enclose one set of model question for Linear Algebra (Math 423)

Model Question

Tribhuvan University
Faculty of Humanities and Social Sciences

Bachelor Level/ II year/Humanities — Full Marks: 70
Mathematics (Math 423) — Pass Marks: 28
Linear Algebra — Time: 3 hours

Candidates are required to give their answers in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt **All** the questions.

Group A $[2 \times 15 = 30]$

1. (a). Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 5 \\ -2 & 3 & 4 \\ 1 & 3 & 6 \end{pmatrix}$ by using elementary row operations. [5]

   (b). Define vector subspace with an example. Prove that the vector sum of two subspaces in a vector space is also a subspace. [2+3]

   OR

   Define basis of a vector space V. Let $A = \begin{pmatrix} 3 & 9 & -12 & 1 \\ 1 & 3 & -4 & 0 \\ 2 & 6 & -8 & 1 \\ 3 & 9 & -12 & 0 \end{pmatrix}$. Find a basis for the null space of A. [1+4]

   (c) Define linear transformation with an example. Find a linear transformation that maps (1,3,-7) to (2,0,2), (3,2,1) to (-1,1,5) and (-3, 5,-23) to (8,-2,5). [2+3]

   OR

   Prove that if T is a linear transformation whose domain is an n-dimensional vector space, then Dim(Ker(T))+ Dim(Range(T))= n. [5]

2. (a) Define eigenvalue of a matrix. Find the eigenvalue of $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. [1+4]

   (b) Define LU factorization. Find the LU factorization of the matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ -1 & -2 & 2 \end{pmatrix}$. [1+4]

   (c) Set up and solve the trafic problem shown:

$$\begin{array}{lllll} 300 \longleftarrow & \overset{x_1}{\uparrow} & \xleftarrow{x_2} & \overset{100}{\uparrow} & \longleftarrow 500 \\ 200 \longrightarrow & \overset{x_3}{\uparrow} & \xrightarrow{x_4} & \overset{x_5}{\uparrow} & \longrightarrow 300 \\ & & & \overset{200}{\uparrow} & \end{array}$$

   . [5]

   OR

   For the set of data that $\{(-3,9),(-2,6),(0,2),(1,1)\}$, use the least squares approximation to find the best fits with a linear function. Also, compute the error E. [5]

Group B $[4 \times 10 = 40]$

3. (a) Examine whether the system of equations x+3y-2z=0, 2x-3y+z=1 and 4x-3y+z= 3 are consistent. If so, solve these equations using Gaussian elimination method. [5]

   (b) Define linearly dependencs of an indexed set of vector space. Prove that if an indexed set of two or more vectors is linearly dependent, then some vectors in the set is a linear combination of the others.

[1+4]

4. For square matrices of the same size,prove that

Det(**AB**)= Det(**A**) Det(**B**). Also, prove that $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x+3a)(x-a)^3.$ [5+5]

OR

Let $A \in M_{n\times n}(F)$, and let B be a matrix obtained by adding a multiple of one row of A to another row of A.Then prove that $|B| = |A|$, and if $A \in M_{n\times n}(F)$ and B is a matrix obtained from A by interchanging any two rows, then prove that $|B| = -|A|$. Also, give an example of each to verify above statements.

[3+3+2+2]

5. Define orthonormal basis of a vector space V. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ generated by the vectors $(1,1,0,0),(1,-1,1,1)$ and $(-1,0,2,1)$. [10]

OR

If U is a finite-dimensional subspace in an inner- product space X, then prove that X is the direct sum of U and $U^\perp$:
X= U + $U^\perp$,
$U \bigcap U^\perp = \{\mathbf{0}\}$,
$U = U^{\perp\perp}$.
Also, if $U = \{(a,b,0) : a,b \in \mathbb{R}\}$ and $W = \{(0,0,c) : c \in \mathbb{R}\}$, show that $\mathbb{R}^3 = U \oplus W$. [7+3]

6. (a) Define symmetric and skew-symmetric matrices. Express the matrix as the sum of a symmetric matrix and a skew- symmetric matrix:
$A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 7 & 5 \\ 6 & 8 & 9 \end{pmatrix}.$ [1+4]
(b) Show that the set of all the real numbers $\mathbb{R}$ form a vector space under the usual operation of addition and scalar multiplication. [5]

4 Math 424 Detailed Course of Differential Equations

TRIBHUVAN UNIVERSITY
Faculty of Humanities and Social Sciences
Micro Syllabus

Course Title: Differential Equations (Compulsory) **Full Marks:** 100
Course No.: Math 424 **Pass Mark:** 40
Level: B.A. **Year:** II
Nature of Course: Theory **Periods:** 9 Lectures Hrs/Week

Course Objectives: This course is designed for the second year of Four Years B.A. Program as a compulsory subject in mathematics. The objective of this course is to acquaint students with the basic concepts of differential equations like first order linear and nonlinear differential equations, second order differential equations and higher order linear equations as well as partial differential equations with their wide range of applications in different fields. It aims at enabling students to build good knowledge base in the subject of ordinary differential equations and partial differential equations.

Detailed Course Contents:

Unit 1 Introduction: [13 Lecture Hrs]

1.1 Some mathematical models and direction fields: Modeling of falling objects, direction field, Idea of constructing mathematical models
Related problems.
1.2 Solutions of differential equations
Related problems.
1.3 Classification of differential equations, Related problems.

Unit 2 : First Order Linear and Nonlinear Differential Equations [18 Lecture Hrs]

2.1 Integrating factors
Related problems.
2.2 separable equations
Related problems.
2.3 Modeling with first order equations
Related problems.
2.4 Difference between Linear and Nonlinear differential equations
Theorem (without proof): If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there is a unique function $y = \phi(t)$ that satisfies the differential equation $y^{'} + p(t)y = g(t)$ for each t in I, and that also satisfies the initial condition $y(t_0) = y_0$.
Theorem (without proof): Let the function f and $\dfrac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $Y < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem $y^{'} = f(t, y), \quad y(t_0) = y_0$.
Related problems.
2.5 Autonomous equations and Population Dynamics (Stability Theory)
Related problems.
2.6 Exact equations and Integrating Factors
Theorem (Statement Only): Let the functions M, N, M_y, N_x, where subscripts denote the partial derivatives, be continuous in the rectangular region $R : \alpha < t < \beta, Y < y < \delta$. Then $M(x, y) + N(x, y)y^{'} = 0$ is an exact differential equation in R if and only if $M_y(x, y) = N_x(x, y)$ at each point on R. That is, there exists a function $\psi_x(x, y) = M(x, y), \quad \psi_y(x, y) = N(x, y)$, if and only if M and N satisfy $M_y(x, y) = N_x(x, y)$.
Related problems.

Unit 3: Numerical Methods [13 Lecture Hrs]

3.1 Numerical Approximations: Eulers method
Related problems.
3.2 Improved Euler's Method
Related problems.
3.3 The Runge-Kutta Method
Related problems.
3.4 System of First order Equations,
Related problems

3.5 First order difference equations
Related problems.

Unit 4: Second Order Linear Equations: [18 Lecture Hrs]

4.1 Homogeneous Equations with constant coefficients.
Related problems
4.2 Solutions of linear homogeneous equations; the Wronskian
Theorem (statement only): Existence and uniqueness theorem for the second order differential equations
Theorem (statement only): Principle of superposition.
Theorem (statement only): Suppose that y_1 and y_2 are two solutions of equation $L[y] = y^{''} + p(t)y^{'} + q(t)y = 0$ with the initial conditions $y(t_0) = y_0, \quad y^{'}(t_0) = y_0^{'}$. Then it is always possible to choose the constants c_1, c_2 so that $y = c_1y_1(t) + c_2y_2(t)$ satisfies the differential equation and the initial conditions if and only if the Wronskian $W = y_1y_2^{'} - y_1^{'}y_2$ is not zero at t_0.
Theorem (statement only): Suppose that y_1 and y_2 are two solutions of equation $L[y] = y^{''} + p(t)y^{'} + q(t)y = 0$. Then the family of solutions $y = c_1y_1(t) + c_2y_2(t)$ with arbitrary coefficients c_1, and c_2 includes every solution if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is not zero.
Theorem (statement only): Consider the differential equation $L[y] = y^{''} + p(t)y^{'} + q(t)y = 0$, whose coefficients p and q are continuous on some open interval I. choose some point t_0 in I. Let y_1 be the solution that satisfies the initial conditions $y(t_0) = 1, \quad y^{'}(t_0) = 0$, and let y_2 be the solution that satisfies the initial conditions $y(t_0) = 0, \quad y^{'}(t_0) = 1$. Then y_1 and y_2 form a fundamental set of solutions.
Theorem (statement and proof): Abel's theorem
Related problems.
4.3 Complex roots of the characteristic equations.
Related problems
4.4 Repeated roots, reduction of order
Related problems
4.5 Non-homogeneous Equations; Method of undetermined coefficients.
Theorem (With proof): If Y_1 and Y_2 are two solutions of the nonhomogeneous equation
L[y] $= y^{''} + p(t)y^{'} + q(t)y = g(t)$, then their difference $Y_1–Y_2$ is a solution of the corresponding homogeneous equation L[y] $= y^{''} + p(t)y^{'} + q(t)y = 0$. If, in addition, y_1 and y_2 are a fundamental set of solutions of the homogeneous equation, then $Y_1(t)–Y_2(t) = c_1y_1(t) + c_2y_2(t)$, where c_1 and c_2 are certain constants.
Theorem (With proof): The general solution of the nonhomogeneous equation
L[y] $= y^{''} + p(t)y^{'} + q(t)y = g(t)$ can be written in the form $y = \phi(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$, where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation , c_1 and c_2 are arbitrary constants, and Y is some specific solution of the nonhomogeneous equation.
Related problems.
4.6 Variation of Parameters
Theorem (statement only): Theorem about particular solution and general solution of a second order nonhomogeneous linear differential equation by using variation of parameters.
Related problems.

Unit 5: Higher Order Linear Equations: [13 Lecture Hrs]

5.1. General Theory of nth order Linear Equations
Theorem (no proof): If the functions $p_1, p_2, \ldots, p_n$ and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation
L[y]$= \dfrac{d^ny}{dt^n} + p_1(t)\dfrac{d^{n-1}y}{dt^{n-1}} + ... + p_{n-1}(t)\dfrac{dy}{dt} + p_n(t)y = g(t)$ that also satisfies the initial conditions
$y(t_0) = y_0, \quad y^{'}(t_0) = y_0^{'}, \quad y^{(n-1)^{'}}(t_0) = y_0^{(n-1)^{'}}$. This solution exists throughout the interval.
Theorem (no proof): If the functions $p_1, p_2, \ldots, p_n$ are continuous on the open interval I, if the functions $y_1, y_2, \ldots, y_n$ are solutions of L[y]$= \dfrac{d^ny}{dt^n} + p_1(t)\dfrac{d^{n-1}y}{dt^{n-1}} + ... + p_{n-1}(t)\dfrac{dy}{dt} + p_n(t)y = 0$ and if $W(y_1, y_2, \ldots, y_n)(t) \neq 0$ for at least one point in I, then every solution can be expressed as a linear combination of the solutions $y_1, y_2, \ldots, y_n$.
Theorem (no proof): If $y_1(t), \ldots, y_n(t)$ is a fundamental set of solutions of on an interval I, then $y_1(t), \ldots, y_n(t)$ are linearly independent on I. Conversely, if $y_1(t), \ldots, y_n(t)$ are linearly independent solutions on I, then they form a fundamental set of solutions on I.
Related problems.
5.2 Homogeneous equations with constant coefficients
Related problems
5.3 Method of undetermined coefficients
Related problems.

5.4 Method of Variation of Parameters
Related problems.

Unit 6: System of First Order Linear Equations: [13 Lecture Hrs]

6.1 Introduction
Theorem (No Proof): Existence and uniqueness theorem for the first order system of differential equations.
Theorem (No Proof): Existence and uniqueness theorem for the first order linear system of differential equations.
Related problems.
6.2. Review of Matrices
No question in exam
6.3 System of Linear algebraic equations: Linear independence, Eigenvalues, Eigenvectors
Related problems
6.4 Basic Theory of first order linear equations
Theorem (No Proof): If the vector functions $x^{(1)}$ and $x^{(2)}$ are the solutions of the system $x^{'} = P(t)x$, then the linear combination $c_1x^{(1)} + c_2x^{(2)}$ is also a solution for any constants c_1 and c_2.
Theorem (No Proof): If the vector functions $x^{(1)}, \ldots, x^{(n)}$ are linearly independent solutions of the system $x^{'} = P(t)x$ on the interval $\alpha < t < \beta$, then in this interval $W[x^{(1)}, \ldots, x^{(n)}]$ either is identically zero or else never vanishes.
Theorem (No Proof): If $x^{(1)}, \ldots, x^{(n)}$ are solutions of the system $x^{'} = P(t)x$ for each point in the interval $\alpha < t < \beta$, then each solution $x = \phi(t)$ of the system can be expressed as a linear combination of $x^{(1)}, \ldots, x^{(n)}\phi(t) = c_1x^{(1)}(t) + \ldots + c_nx^{(n)}(t)$ in exactly one way.
Theorem (No Proof): Let $e^{(1)} = (1, 0, 0, \ldots, 0)^T$, $e^{(2)} = (0, 1, 0, \ldots, 0)^T, \ldots,$ $e^{(n)} = (0, 0, 0, \ldots, 1)^T$. Further let $x^{(1)}, \ldots, x^{(n)}$, be the solutions of the system $x^{'} = P(t)x$ that satisfies the initial conditions $x^{(1)}(t_0) = e^{(1)}, \quad \ldots, \quad x^{(n)}(t_0) = e^{(n)}$ respectively, where t is any point in $\alpha < t < \beta$. Then $x^{(1)}, \ldots, x^{(n)}$ form a fundamental solutions of the system $x^{'} = P(t)x$.
Related problems.
6.5 Homogeneous Linear Systems with Constant Coefficients
Related problems.

Unit 7: Laplace Transforms and Their Applications to Differential Equations [13 Lecture Hrs]

7.1 Definition and Examples of Laplace Transforms
7.2 Properties of Laplace Transform; Linearity, Shifting (Translation)
7.3 The Laplace Transform of some basic functions like; $1, e^{at}, t^n, \sin at, \cos bt, \sinh kt$ etc.
7.4 The Inverse Laplace Transform and Convolution
7.5 Transform of Derivatives,
7.6 Transforms of Integrals
7.7 Unit Step Function
7.8 Solution of a Linear Differential Equations with Constant Coefficients Using Laplace Transform Methods.
Problems of finding Laplace transform and Inverse Laplace Transform of some functions, Problems related to the properties, Problems of solving linear differential equations with constant coefficients including initial value problems.

Unit 8: Introduction to Partial Differential Equations [18 Lecture Hrs]

8.1 Introduction to Partial Differential Equations
8.2 Formation of PDEs by elimination of arbitrary constants and arbitrary functions
with some examples and related problems
8.3 Solutions of PDEs with examples and related problems
8.4 Equations Easily Integrable with some examples and related problems
8.5 Linear Equations of the First Order: Lagrange's Linear PDEs with some examples and related problems.
8.6 Nonlinear PDEs of First Order: Charpit's Method (Complete and particular integrals, General Integrals, Charpit's Method with examples and related problems).

Unit 9: Partial Differential Equations and Fourier Series [18 Lecture Hrs]

9.1 Two point boundary value Problems
Related problems.
9.2 Fourier series
Related problems.
9.3 The Fourier Convergence Theorem
Theorem (No Proof): Fourier Convergence theorem
Related problems

9.4 Even and odd functions
Related problems.

Unit 10: Separation of Variables [13 Lecture Hrs]
10.1 Separation of variables; Heat conduction in a Rod
Related problems
10.2 Other heat conduction Problems
Related problems
10.3 The wave equation: Vibration of an Elastic string
Related problems
10.4 Laplace's equations
Related problems

Text/ Reference Books:

1. Boyce, W. and DiPrima, R.: *Elementary Differential Equations and Boundary Value Problems*, 9th Ed., Wiley India.
2. Zafar Ahsan: *Differential Equations and Their Applications*, Second Edition, PHI Learning Pvt. Ltd.
3. Durga Jang K.C. and Tek Bahadur Budhathoki: *A Textbook of Differential Equations*, Heritage Publishers & Distributors Pvt. Ltd.
4. James C. Robinson: *An Introduction to Ordinary Differential Equations*, Cambridge University Press.
5. P. Kattel and J. Kafle: *Differential Equations with Applications*, Sukunda Pustak Bhawan.

Examination:
There will be a final examination of 70 marks for the period of three hours. The internal examination of 30 marks will be conducted by the department of mathematics of related campus and the marks will be submitted to Tribhuvan University Office of the Controller of Examination, Balkhu. A candidate must pass the internal and the final examinations separately.

Marks allocation for the internal examination:
- Written examinations: 20 marks (1 hour)
- A student or a group of students with presentation: 5 marks
- Assignments: 5 marks

Guidelines to the question setter:

In the final examination
1. Questions must include every unit.
2. There will be two groups, namely, Group A and Group B.
3. In group A, there must be OR selection for 15 marks questions.
4. In group B , there must be OR selection for 20 marks questions.
5. OR Selection must be given from the same unit.
6. Questions must be creative and should be appropriate to the allocated time.

On the basis of the guidelines mentioned, we enclose one set of model question for Differential Equations (Math 424)

Model Question

Tribhuvan University
Faculty of Humanities and Social Sciences

Bachelor Level/ II year/Humanities — Full Marks: 70
Mathematics (Math 424) — Pass Marks: 28
Differential Equations — Time: 3 hours

Candidates are required to give their answers in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt **All** the questions.

Group A [$2 \times 15 = 30$]

1. (a) Define solution of a differential equation. Determine the value of r for which the differential equation $t^2y^{''} + 4ty^{'} + 2y = 0$ has solutions of the form $y = t^r$ for $t > 0$. [1+4]

   (b) For the system $x_1{}^{'} = -2x_1 + x_2, \quad x_2{}^{'} = x_1 - 2x_2$, find the solution that satisfies the initial conditions $x_1(0) = 2$ and $x_2(0) = 3$. [5]

   OR

   Find the Wronskian of the vectors $\mathbf{X}^{(1)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ and $\mathbf{X}^{(2)}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$. What conclusion can be drawn about the coefficients in the system of homogeneous differential equations $\mathbf{X}^{'} = \mathbf{P}(t)\mathbf{X}$ satisfied by $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$? [2+3]

   (c) Find the solution of the initial value problem

$$y^{(4)} - y = 0, \quad y(0) = \frac{7}{2}, \quad y^{'}(0) = -4, \quad y^{''}(0) = \frac{5}{2}, \quad y^{'''}(0) = -2$$

   . [5]

2. (a) Define boundary conditions of a differential equation. Find the steady state solution of the equation $\alpha^2 u_{xx} = u_t$ that satisfies the boundary conditions $u(0, t) = T, \ u_x(L, t) = 0$. [1+4]

   OR

   Find the general solution of one-dimensional wave equation $a^2 u_{xx} = u_{tt}$. [5]

   (b) Define equilibrium solution of a difference equation. Solve $y_{n+1} = \frac{n+1}{n+2}\, y_n$ in terms of the initial value y_0 and also describe the behaviour of the solution as $n \to \infty$. [1+4]

   OR

   Use the Euler's method to show that the numerical solution of the initial value problem $y^{'} = 2y - 1; \quad y(0) = 1$ converges to $\frac{1}{2}e^{2t} + \frac{1}{2}$. [5]

   (c) By using the Laplace transform, solve: $y^{''} + 3y^{'} + 2y = 12e^{2x}$. [5]

Group B [$4 \times 10 = 40$]

3. Suppose a brine containing 0.2 kg of salt per liter runs into a tank initially filled with 500L of water containing 5 kg of salt. The brine enters the tank at a rate of 5L/min. The mixture, kept uniform by stirring, is flowing out at the rate of 5L/min.

   (a) Write the appropriate variables with their units. [2]

   (b) Construct a mathematical model of this flow process, that is, find the differential equation that describes this process. [3]

   (c) By solving the differential equation obtained in part (b), find the concentration of the salt in the tank after 10 minutes. [5]

   OR

   (a) Solve the initial value problem $y^{'} = y^2, \quad y(0) = 1$, and determine the interval in which the solution exists. [3+1]

(b) For the differential equation $dy/dt = y(y-1)(y-2)$, sketch the graph of $dy/dt = f(y)$ versus y, determine the critical (equilibrium) points and classify each one as asymptotically stable, semistable or unstable. Draw the phase line and sketch several graphs of solutions in the ty-plane. [6]

4. (a) Verify that $y = 1$ and $y = t^{\frac{1}{2}}$ are solutions of the differential equation $yy^{''} + y^{'2} = 0$ for $t > 0$. Then show that $y = 1 + t^{\frac{1}{2}}$ is not a solution. Explain why this does not contradict the existence and uniqueness theorem or the principle of superposition. [2+1+1]

   (b) Find the Wronskian of the functions $y = \cos^2 t$ and $y = 1 + \cos 2t$. Can these two functions form a fundamental set of solutions for second order differential equations? Justify your answer. [2+1]

   (c) Without solving the problem, determine an interval in which the solution of the initial value problem; $y^{'} + \tan t\ y = \sin t, \quad y(\pi) = 0$ is certain to exist. [3]

5. Define linear and nonlinear partial differential equation with an example of each.
Solve: $(y + z)p + (z + x)q = x + y$, where the symbols have there usual meanings. [3+7]

6. Assume that the function $f(x)$ defined by

$$f(x) = \begin{cases} -1, & -1 \leq x < 0 \\ 1, & 0 \leq x < 1 \end{cases}$$

is periodically extended outside the original interval. Find the Fourier series for the extended function.

[10]

OR

Find the solution of the initial value problem $y^{''} + \omega^2 y = \sin nt, \quad y(0) = 0, \quad y^{'}(0) = 0$, where n is a positive integer and $\omega^2 \neq n^2$. What happens if $\omega^2 = n^2$? [8+2]

5 Math 425 Detailed Course of Real Analysis

TRIBHUVAN UNIVERSITY
Faculty of Humanities and Social Sciences
Micro Syllabus

Course Title: Real Analysis (Compulsory)	**Full Marks:** 100
Course No.: Math 425	**Pass Marks:** 40
Level: B.A.	**Year:** III
Nature of Course: Theory	**Periods:** 9 Lecture Hrs/Week

Course Objectives: This course is designed for the third year of Four Years B.A. Program as a compulsory subject in mathematics. The general objectives of this course is to acquire basic knowledge and understanding of the language of mathematical terms, symbols, statements, formulae, definitions, logic, sets etc. Also, the course aims to develop basic knowledge and analytical skill in the emerging areas of Real Analysis and to prepare a base for higher studies in Mathematical Analysis.

Detailed Course Contents:

Unit 1. Logic, Sets and Functions [13 Lecture Hrs]

- Sentence and statement.
- Connectives and compound statements.
- Tautology and contradiction.
- Inference and proof.
- Techniques of proof.
- Quantifiers.
- Sets and set operations.
- Laws of set algebra.
- Classes of sets.
- Relations.
- Equivalence relations.
- Functions.
- Composition of functions and inverse of functions.
- Related problems.

Unit 2. Real Number System [18 Lecture Hrs]

- Peano's axioms, Field axioms, Order axioms, Absolute value and distance, Completeness axioms.
- Properties of supremum and infimum: Uniqueness, Approximation property, Additive property, Comparison property, Archimedean property, Well-ordering principle, Rational density theorem , Irrational density theorem.
- Geometrical representation of the real numbers, Cardinality, Denumerable, Countable and uncountable sets.
- Theorem : Every subset of a countable set is countable.
- Theorem: If $F = \{A_i : i = 1, 2, 3, \cdots\}$ is a countable collection of countable sets then the union $\cup A_i$ of all sets in F is also countable.
- Theorem: The complement of a countable subset of an uncountable set is also uncountable.
- Countable and uncountable subsets of the set of real numbers.
- Theorem: Every subset of the set of positive integers is countable.
- Theorem: The set $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countably infinite.

- Theorem: The set of all rational numbers is countable.
- Theorem: The set of all infinite sequences of natural numbers is uncountable.
- Theorem: The set of all real numbers between 0 and 1 is not countable.
- Theorem: There exists a one to one correspondence between the open interval $(0, 1)$ and the real line $\mathbb{R}$.
- Related problems.

Unit 3. Point-Set Topology of the Real Line [18 Lecture Hrs]

- Introduction, Intervals, Neighbourhood and interior points, Open sets in $\mathbb{R}$, Properties of open sets, Closed sets, Properties of closed sets.
- Adherent points and closure of a set.
- Limit points and derived set of a set.
- Boundary points and boundary of a set.
- Theorem: A set S in $\mathbb{R}$ is closed if, and only if, it contains all its adherent points.
- Theorem: A set S is closed if, and only if, $S = \overline{S}$.
- Theorem: Let S be a set in R and S' its derived set. Then its closure $\overline{S} = S \cup S'$.
- Theorem: If c is a limit point of a set S in R , then every open interval $B(c; r)$ contains infinitely many points of S.
- Theorem: A set S in $\mathbb{R}$ is closed if, and only if, it contains all its limit points.
- Theorem: A set S in $\mathbb{R}$ is closed if, and only if, it contains all its boundary points.
- Theorem: Every bounded infinite set in $\mathbb{R}$ has a limit point in $\mathbb{R}$.
- Nested interval theorem.
- Concept of perfect set and Cantor set.
- Related problems.

Unit 4. Sequences of Real Numbers [18 Lecture Hrs.]

- Infinite sequence, Sequence of real numbers, Monotonic and bounded sequences, Subsequence.
- Convergent sequences.
- Theorem: A sequence of real numbers can have at most one limit.
- Theorem: A convergent sequence of real numbers is bounded.
- Theorem: A real sequence $\{x_n\}$ converges to x if and only if given $\epsilon > 0$, there exists a positive integer N such that for all $n \geq N$, $|x_n - x| < M\epsilon$, where $M > 0$ is independent of both N and ϵ.
- Theorem: Let $y_n \to 0$. If for some $M > 0$, there exists a positive integer N such that for all $n \geq N$,$|x_n - x| \leq M|y_n|$ then $x_n \to x$.
- Theorem: A real number x is an adherent point of a set S in $\mathbb{R}$ if and only if there is a sequence $\{x_n\}$ in S converging to x.
- Operations on convergent sequences.
- Order properties.
- Simple convergence criteria
  - Sandwich theorem.
  - Monotone convergence criterion.
- Theorem: If the sequence $\{x_n\}$ is monotonically decreasing or non-increasing and is bounded below then it is convergent and attains its greatest lower bound. Analogously for monotonically increasing sequence.
- Theorem: If a real sequence converges, then it is bounded.
- Relations between sequences and subsequences.
  - Forced limit.
  - Bolzano - Weierstrass theorem.
  - Nested intervals theorem for sequences.

  - Cauchy sequences.
  - Properties of Cauchy sequences.
- Concept of limit superior and limit inferior of a bounded sequence.
- Related problems.

Unit 5. Series of Real Numbers [13 Lecture Hrs]

- Series
- Infinite series of real numbers.
- Convergence and divergence of an infinite series.
- Cauchy's criterion of convergence for infinite series.
- Series of non-negative terms, p-series test.
- Alternating series, Leibnitz test for alternating series.
- Different tests for convergence
  - Direct comparison test.
  - Limit comparison test.
  - D'Alembert's ratio test.
  - Cauchy's root test.
- Absolute and conditional convergence.
- Raabe's test (No proof).
- Logarithmic ratio test (No proof).
- Related problems

Unit 6. Limit of a Function [13 Lecture Hrs]

- Limit of a function with different approaches.
- Limit at infinity.
- Infinite limit.
- Algebraic properties of limits.
- Uniqueness of limit.
- Squeezing theorem.
- Sequential criteria for limit of a function.
- One sided limits with geometry.
- Related problems.

Unit 7. Continuity of Function [13 Lecture Hrs]

- Continuous function.
- One sided continuity.
- Algebra of continuous functions.
- Continuity of composite functions.
- Continuity of polynomial function.
- Discontinuities and its types with examples.
- Continuity and inverse images of open and closed sets.
- Sequential criterion for continuity.
- Continuity and compact sets.
- Image of a compact set under a continuous function.
- Continuous function on a closed and bounded set.
- Sign preserving theorem.

- Bolzano theorem.
- Intermediate value theorem
- Uniform continuity and boundedness
- Lipschitz's condition
- Related problems

Unit 8. Differentiation [18 Lecture Hrs]

- Derivative of a real valued function of a single real variable.
- Rules of differentiation.
- Sequential criterion for derivative.
- Differentiability and continuity.
- Differential
- Rolle's theorem.
- Mean value theorem.
- Test for constancy of a function.
- Cauchy's mean value theorem.
- Indeterminate forms and L' Hospital's rule.
- Test for monotonocity of a function.
- Extreme values.
- Stationary point and critical point.
- Point of inflection and concavity.
- Taylor's theorem and applications.
- Related problems.

Unit 9. Riemann Integration [13 Lecture Hrs]

- Partitions and norm, refinement
- Riemann integral of bounded functions
- Darboux / Riemann sums
- Riemann integral and Riemann integrable function
- Relation between upper and lower sums
- Relation between lower and upper integrals
- Riemann's conditions of integrability
- Sufficient condition for integrability
- Elementary properties of Riemann integrals
- Integrals of step function
- Properties of integrals of step functions
- Related problems

Unit 10. Fundamental Theorems of Integral Calculus [13 Lecture Hrs]

- Primitive of a function
- First mean value theorem for Riemann integral
- Generalized first mean value theorem for integrals
- First fundamental theorem of integral calculus
- Second fundamental theorem of integral calculus
- Integration by parts

- Change of variable in an integral
- Second mean value theorem for integrals
- Generalized second mean value theorem for Riemann integrals
- Related problems

Text/ Reference Books:

1. Bartle, Robert G. & Sherbert, Donald R.: *Introduction to Real Analysis*, John Wiley and Sons Inc., Singapore.
2. Bajracharya, P.M.: *Real Analysis, An Introduction to Proof*, Buddha Academic Publishers & Distributors Pvt.Ltd., Kathmandu, Nepal
3. Shrestha, R.M. and Pahari,N.P.: *Fundamentals of Mathematical Analysis*, Sukunda Pustak Bhawan, Kathmandu, Nepal.
4. Malik, S.C.& Arora, Savita: *Mathematical Analysis*, Wiley Eastern Limited, New Delhi.
5. Gupta , S.L. & Rani, Nisa: *Fundamental Real Analysis*, Vikas Publishing House, New Delhi.

Examination:
There will be a final examination of 70 marks for the period of three hours. The internal examination of 30 marks will be conducted by the department of mathematics of related campus and the marks will be submitted to Tribhuvan University Office of the Controller of Examination, Balkhu. A candidate must pass the internal and the final examinations separately.
Marks allocation for the internal examination:

- Written examinations: 20 marks (1 hour)
- A student or a group of students with presentation: 5 marks
- Assignments: 5 marks

Guidelines to the question setter:

In the final examination
1. Questions must include every unit.
2. There will be two groups, namely, Group A and Group B.
3. In group A, there must be OR selection for 15 marks questions.
4. In group B , there must be OR selection for 20 marks questions.
5. OR Selection must be given from the same unit.
6. Questions must be creative and should be appropriate to the allocated time.

On the basis of the guidelines mentioned, we enclose one set of model question for Real Analysis (Math 425)

Model Question

Tribhuvan University
Faculty of Humanities and Social Sciences

Bachelor Level/ III year/Humanities — Full Marks: 70
Mathematics (Math 425) — Pass Marks: 28
Real Analysis — Time: 3 hours

Candidates are required to give their answers in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt **All** the questions.

Group A [$2 \times 15 = 30$]

1. (a) Let $f : X \to Y$ and $g : Y \to Z$ be two bijective functions. Show that $g \circ f : X \to Z$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. [5]

   (b) State limit comparison test for the convergence of an infinite series.Test the convergence of the series $1 + \dfrac{2}{1^3+2} + \dfrac{5}{2^3+2} + \dfrac{10}{3^3+2} + \cdots$. [1+4]

   OR

   Prove that the series $\sum_{n=1}^{\infty} n^{-p}$ converges for $p > 1$ and diverges for $p \leq 1$. [5]

   (c) Define limit of a function at a point. Prove that the limit of a function, if it exists then it is unique. [1+4]

2. (a) Distinguish between continuity and uniform continuity of a function of a real variable. Discuss the continuity and uniform continuity the function $f(x) = x^2$ on $\mathbb{R}$. [1+4]

   OR

   Define compact set. Let a real valued function of a real variable $f : \mathbb{R} \to \mathbb{R}$ be continuous on a compact subset S of the domain $\mathbb{R}$. Prove that the image set $f(S)$ is compact in $\mathbb{R}$. [1+4]

   (b) Show that the greatest integer function $f(x) = [x]$ is integrable on $[0, 4]$ and $\int_0^4 [x]dx = 6$. [5]

   OR

   State Riemann's condition of integrability. Prove that if f and g are integrable on $[a, b]$ then fg is integrable on $[a, b]$. [1+4]

   (c) Distinguish between primitive and integral. State and prove first fundamental theorem of integral calculus. [1+4]

Group B [$4 \times 10 = 40$]

3. Define supremum and infimum for a bounded set with one example of each. Let $S = \left\{ x : x = \dfrac{n}{n+1}, n \in \mathbb{Z}^+ \right\}$. Show that S is bounded and l.u.b $S = 1$ and g.l.b. $S = \dfrac{1}{2}$. [2+2+6]

   OR

   Define countable set. Prove that the set of rational numbers is countable. Also, prove that the set of all real numbers between 0 and 1 is not countable. [1+4+5]

4. (a) Prove that a set S in $\mathbb{R}$ is closed if, and only if, it contains all its adherent points.

   (b) State the necessary condition for a set in $\mathbb{R}$ to have a limit point. Prove that every bounded infinite set in $\mathbb{R}$ has a limit point in $\mathbb{R}$. [4+6]

5. (a) Define Cauchy sequence. Prove that every Cauchy sequence in $\mathbb{R}$ is bounded. [1+4]

   (b) Define a convergent sequence. Discuss the convergence of the sequence $\{x_n\}$ given by $x_n = \left(1 + \frac{1}{n}\right)^n$. [1+4]

6. (a) State Taylor's series for real valued continuous function. Use a Taylor's polynomial to approximate $f(x) = \dfrac{1}{1+x}$ at $x = 0.1$ with error less than 0.001. [7]

(b) Prove or disprove that if f on $(a, b) \subset \mathbb{R}$ is continuous at c in (a, b) then f is differentiable at c. [3]

OR

(a) State and prove Rolle's theorem. Give its geometrical meaning. Verify Rolle's theorem for $f(x) = \sin x$ in $[0, \pi]$. [4+2+2]

(b) Define point of inflection of a function. Find the point of inflection of $y = x^4$ if it exists. [2]

6 Math 410A Detailed Course of Linear Programming and Discrete Mathematics

TRIBHUVAN UNIVERSITY
Faculty of Humanities and Social Sciences
Micro Syllabus

Course Title: Linear Programming and Discrete Mathematics (Elective)	**Full Marks:** 100
Course No.: Math 410A	**Pass Marks:** 40
Level: B.A.	**Year:** III
Nature of Course: Theory	**Periods:** 9 Hrs/Week

Course Objectives: This course is designed for the third year of Four Years B.A. Program as an elective subject. The course covers basic introduction of linear programming problems, their solution techniques and implementation of the solution techniques to solve real world problems formulated as linear programming problem. After the completion of this course, students will be able to

- know the concept and importance of convexity in optimization.
- formulate instances of real world problems in the form of linear programming problem (LPP).
- know solution techniques for the solution of LPP.
- understand the notions of graphs and some algorithms.
- familiarize with some applications.

Detailed Course Contents:

Unit 1. Linearity and Convexity [21 Lecture Hrs]

Basic definitions, concepts and examples of

- system of linear equations and inequalities
- matrix representation of system of linear equations and inequalities
- linear dependence and independence
- basic solutions
- lines and hyper planes
- affine and convex sets
- convex cones, polyhedra
- convex functions
- basic properties of convex functions
  - first-order conditions with proof
  - second-order conditions
- tutorial on related problems

Note: Resource material- **Text/Reference Book(s):** 1

Unit 2. Linear Programming (LP)-Models [11 Lecture Hrs]

- variables and constraints
- cost function
- formulation strategy of an LP problem
- general, canonical and standard forms of LP models
- slack and surplus variables
- the equivalent relation of different LP-forms with proof
- LP problem in matrix notation

- formulation of some instances of real-world problems in the form of LP-models
  - feed mix problem
  - production scheduling
  - cutting stock problem
  - the transportation problem
  - capital budgeting problem
- tutorial on related problems.

Note: Resource material - **Text/Reference Book(s):** 1

Unit 3. Solution Methods to LP Problems [11 Lecture Hrs]

- extreme points and optimality
- basic feasible solutions (bfs)
  - degenerate bfs
  - correspondance with bfs and extreme points
  - existence of bfs
  - theorems 3.1, 3.2, 3.3 and 3.4 (no proof)
- solution of LP with two variables: graphical method
- key simplex method
- Tutorial on related problems.

Note: Resource material - **Text/Reference Book(s):** 1

Unit 4. Simplex Method [21 Lecture Hrs]

- geometric motivation of the simplex method
- algebra of the simplex method
  - interpretation of entering and leaving basis
  - blocking variable
- termination: optimality and un-boundedness
  - termination with an optimal solution
  - unique and alternative optimal solutions
  - unboundedness
- the simplex Method
  - initialization step
  - main step
  - modification for a maximization problem
  - finite convergence of the simplex method in the absence of degeneracy
  - theorem 3.5 Finite convergence (no proof)
- the simplex method in tableau format
  - pivoting
  - initialization step
  - main step
  - interpretation of entries in the simplex tableau
  - indentifying B^{-1}
- block pivoting
- the simplex tableau and examples
- tutorial on related problems.

Note: Resource material- **Text/Reference Book(s):** 1

Unit 5. LP Duality Theory [11 Lecture Hrs]

- the dual LP formulation
  - canonical form of duality
  - standard form of duality
  - dual of the dual
  - lemma 6.1 (no proof)
  - mixed form of duality
  - primal-dual relationships
  - weak duality property
  - fundamental theorem of duality (no proof)
- complementary slackness conditions
  - complementary slackness theorem
- The dual simplex algorithm
  - interpretation of dual feasibility on the primal simplex tableau
  - the dual simplex method
- tutorial on related problems.

Note: Resource material- **Text/Reference Book(s):** 1

Unit 6. Graphs [11 Lecture Hrs]

- notation and definitions
  - basic nottions, subgraphs, factors, digraphs
  - lemma 1.1.1
  -
- path, cycles, connectedness, tree(basic)
  - lemma 1.2.6
  - lemma 1.2.7
  - theorem 1.2.8
  - theorem 1.2.10
  - corollary 1.2.11
  - lemma 1.2.12
  - theorem 1.2.13 (no proof)
- Eulerian and Hamiltonian Graph
  - definitions
  - theorem 1.3.1 (Euler's theorem
  - corollary 1.3.2
  - theorem 1.4.1
  - corollary 1.4.2
  - corollary 1.4.3
- tutorial on related problems.

Note: Resource material- **Text/Reference Book(s):** 3

Unit 7. Trees [11 Lecture Hrs]

- introduction to trees
- tree traversal, spanning trees
  - theorem 9.1
  - corlollary 9.2
  - theorem 9.3

- applications of trees
  - the minimum connector problem
  - enumeration of chemical molecules
  - electrical networks
  - searching trees
- tutorial on related problems.

Note: Resource material- **Text/Reference Book(s):** 2

Unit 8. Shortest Paths [21 Lecture Hrs]

- shortest paths
- finite metric spaces
  - definition
  - lemma 3.2.1
  - proposition 3.2.2
- breadth first search and bipartite graphs
  - algorithm 3.3.1
  - theorem 3.3.2
  - corollary 3.3.3 (no proof)
  - theorem 3.3.5
  - algorithm 3.3.6
  - theorem 3.3.7 (no proof)
- the algorithm of Dijkstra
  - algorithm 3.6.1
  - theorem 3.6.2
  - algorithm 3.6.6
  - theorem 3.6.7 (no proof)
- tutorial on related problems.

Note: Resource material- **Text/Reference Book(s):** 3

Unit 9. Spanning Trees [21 Lecture Hrs]

- trees and forests
  - lemma 4.1.1
- incidence matrices
  - definition 4.2.1
  - lemma 4.2.2
  - theorem 4.2.3
  - theorem 4.2.4
  - theorem 4.2.5
  - corollary 4.2.6
  - theorem 4.2.7
  - lemma 4.2.8
  - theorem 4.2.9
- minimal spanning trees
  - theorem 4.3.1
  - lemma 4.3.2
  - theorem 4.3.3
- the algorithms of Prim

  - algorithm 4.4.1
  - theorem 4.4.2 (no proof)
  - algorithm 4.4.3
- Kruskal's and Boruvka's algorithms
  - algorithm 4.4.6
  - algorithm 4.4.7
  - algorithm 4.4.10
- tutorial on related problems.

Note: Resource material- **Text/Reference Book(s):** 3

Unit 10. Application [11 Lecture Hrs]

- assignment problem
- maximal flows in a network
- minimum cost flow problem
- transshipment problem
- diet problem
- traveling salesperson problem
- transportation problem
- tutorial on related problems.

Note: Resource material- **Text/Reference Book(s):** 3.
Instructors are advised to provide some aforementioned problems to the students as an assignment.

Text/Reference Books:

1. M. S. Bazaraa, J. J. Jarvis, H. D.Sherali, *Linear Programming and Network Flows*,John Wiley & Sons, Inc., Hoboken, New Jersey 2010
2. Robin J. Wilson, *Introduction to Graph Theory*, Addison Wesley Longman Limited.
3. D. Jungnickel, *Graphs, Networks and Algorithms*, Springer.

Note: Some contents of the syllabus may not be found in the prescribed resource materials. One may search such contents in any useful resource material.
Examination:
There will be a final examination of 70 marks for the period of three hours. The internal examination of 30 marks will be conducted by the department of mathematics of related campus and the marks will be submitted to Tribhuvan University Office of the Controller of Examination, Balkhu. A candidate must pass the internal and the final examinations separately.
Marks allocation for the internal examination:

- Written examinations: 20 marks (1 hour)
- A student or a group of students with presentation: 5 marks
- Assignments: 5 marks

Guidelines to the question setter:
In the final examination
1. Questions must include every unit.
2. There will be two groups, namely, Group A and Group B.
3. In group A, there must be OR selection for 15 marks questions.
4. In group B , there must be OR selection for 20 marks questions.
5. OR Selection must be given from the same unit.
6. Questions must be creative and should be appropriate to the allocated time.

On the basis of the guidelines mentioned, we enclose one set of model question for Linear Programming and Discrete Mathematics (Math 410 A)

Model Question

Tribhuvan University
Faculty of Humanities and Social Sciences

Bachelor Level/ III Year / Humanities — Full Marks:70
Mathematics (Math 410A) — Pass Mark:28
Linear Programming and Discrete Mathematics — Time:3 Hours

Candidates are required to give their answers in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt **All** the questions.

Group A [$2 \times 15 = 30$]

1. (a) Define a convex set and verify that $\{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ is a convex set. [1+4]

OR

State and prove the first-order condition of a convex function. [1+4]

(b) What strategy is to be followed while formulating an LP model ? Use the strategy and formulate the transportation problem in LP model. [1+4]

(c) Solve the following with graphical method [5]

$$\begin{array}{rll} \textit{maximize} & x_1 + 3x_2 & \\ \textit{s.t.} & x_1 - 2x_2 & \leq 0 \\ & -2x_1 + x_2 & \leq 4 \\ & 5x_1 + 3x_2 & \leq 15 \\ & x_1, x_2 \geq 0 & \end{array}$$

2. (a) Show that any finite metric space can be realized by a network with a positive length function. [5]

(b) Describe the breath first search algorithm. [5]

OR

Define a tree and a forest. [1+1+8]
Let T be a graph with n vertices. Then the following statements are equivalent:
(i) T is a tree
(ii) T contains no cycles, and has n—1 edges
(iii) T is connected, and has n-1 edges
(iv) T is connected, and each edge is a bridge
(v) any two vertices of T are connected by exactly one path
(vi) T contains no cycles, but the addition of any new edge creates exactly one cycle.

(c) Let M be the incidence matrix of a digraph G. Then prove that M is totally unimodular. [5]

Group B [$4 \times 10 = 40$]

3. Solve the following using simplex method: [10]

$$\begin{array}{rll} \textit{minimize} & -x_1 - 3x_2 & \\ \textit{s.t.} & 2x_1 + 3x_2 & \leq 6 \\ & -x_1 + x_2 & \leq 1 \\ & x_1, x_2 \geq 0. & \end{array}$$

4. (a) State and prove the complementary slackness theorem. [5]

OR

Write a procedure that converts an LP primal problem into dual form. Dualize the following primal LP problem. [3+2]

$$\begin{array}{rll} \textit{minimize} & -x_1 - 3x_2 & \\ s.t. & 2x_1 + 3x_2 & \leq 6 \\ & -x_1 + x_2 & \leq 1 \\ & x_1, x_2 \geq 0. & \end{array}$$

(b) Let G be a graph with n vertices. Then prove that any two of the following conditions imply the third: (a) G is connected. (b) G is acyclic. (c) G has n-1 edges. [5]

OR

Define an incidence matrix as a representation of a digraph. Moreover consider an appropriate digraph $G = (V, A)$ with $|V| = 5$ and $|A| = 7$ and represent it with the incidence matrix. [1+4]

5. Describe how a maximum flow problem is modeled over a graph in LP form. [10]

OR

Describe how a minimum cost flow problem is modeled over a graph in LP form. [10]

6. Discuss an algorithm which obtains minimal spanning tree and assure it with a formal proof. [5+5]

7 Math 410B Detailed Course of Computer Programming

TRIBHUVAN UNIVERSITY
Faculty of Humanities and Social Sciences
Micro Syllabus

Course Title: Computer Programming (Elective) **Full Marks:** 100 (50 Theory + 50 Lab)
Course No.: Math 410B **Pass Mark:** 40 (20 Theory + 20 Lab)
Level: B.A. **Year:** III
Nature of Course: Theory and Practical **Periods:** 9 Lecture hours/Week

Course Objectives: This course is designed for the third year of Four Years B.A. Program as an elective subject. The course aims at providing exposure to mathematical problem-solving through computer programming. With the help of programming in C, it provides basic constructs for most of the high-level programming languages. After the completion of the course, the students will be able to develop logic so that they can write computer programs focusing on the various problems related to mathematics courses.

Detailed Course Contents

Part A: Theory **[75 Lecture Hrs]**

Unit 1: Introduction to C Programming [11 Lecture Hrs]

- Computer languages
  - Machine language
  - High-level language
  - Compilation, interpretation
- Introduction to C
- Importance of C
- Basic structure of a C program
- Compilation and execution of a C program using an IDE
- The C-character set: Letters, numbers, special characters, white spaces
- C-tokens
  - Keywords and identifiers
  - Delimiters: comma, semicolon, quotes, braces, pipes, slashes
- Data types
  - Basic data types: `int, char, float, double`
  - Qualifiers: `short, long, signed, unsigned`
  - Decimal, binary, octal, hexadecimal integers
  - Type declarations
- Constants
  - Primary and secondary constants
  - Rules for constructing integer, real, character constants
- Variables
  - Definition
  - Declaration of a variable
  - Rules of constructing variable names
- Expressions, Statements, Comments, Symbolic constants
- Different types of errors
  - Run-time errors
  - Compile errors
  - Logical errors
- Debugging of programs

Unit 2. C Instructions [21 Lecture Hrs]

- Input-Output Statements
  - Single character input-output
    - The `getchar` function
    - The `putchar` function
  - Input data using `scanf`
    - General form of `scanf`
    - Format specifiers or Conversion characters
    - Multiple inputs using white spaces
    - Specifying maximum field width
  - Writing output data using `printf`
    - General form of `printf`
    - Conversion characters
    - Escape sequences
    - Reading and writing a line of text
    - Specifying minimum field width
    - Use of `%g`
    - Use of flags
  - `gets` and `puts` functions
- Operators and Expressions
  - Overview of operators and expressions (in relation to operators and expressions in mathematics)
  - Arithmetic operators
    - Addition, subtraction, multiplication, division, modulo operators
    - Integer division vs float division
    - Exponentiation using the library function `pow`
    - Arithmetic expressions
  - Unary operators
    - Unary minus
    - The increment operator `++`
    - The decrement operator `--`
  - Relational operators
    - Less than ($<$), less than or equal to ($<=$), greater than ($>$), greater than or equal to $>=$ operators
  - Comparison operators: is equal to ($==$) and is not equal to ($!=$)
  - Logical operators
    - Logical AND (`&&`) and OR ($||$) operators
    - Logical operators in combination with other operators
  - Assignment Operators
    - The assignment operator $=$
    - The assignment operators `+=`, `-=`, `*=`, `/=`, `%=`
  - The conditional operator (`?:`)
  - Bitwise operators
    - Bit numbering and conversion
    - One's complement operator
    - Right shift operator
    - Left shift operator
    - Bitwise AND operator
    - Bitwise OR operator
    - Bitwise XOR operator
  - Comma operator
  - Precedence of operators in C expressions
- Type conversion in expressions

- Programming examples using the operators, e.g., calculation of simple interest, compound interest, area of a circle

Unit 3. Decision Control Instructions [11 Lecture Hrs]

- Sequential vs. controlled execution of instructions
- Branching
  - The `if` statement
    - General form
    - Use of logical operators with `if`
    - Single and multiple statements within `if`
  - The `if-else` statement
    - General form
    - Conditional operator vs. `if-else`
    - Nested `if-else`s
  - Programming examples using `if-else`, e.g., finding absolute value, testing whether a number lies in a given interval (open, closed), finding roots of a quadratic equation
- Looping
  - Iterative process
  - The `while` and `do-while` statements
    - General forms
    - Difference between `while` and `do-while`
    - Programming examples using `while`, `do-while`, e.g., generating a sequence, summing up a series, finding arithmetic mean, calculating factorial
  - The `for` statement
    - General form
    - for with comma operator
    - Programming examples using for, e.g., generating a sequence, summing up a series, finding arithmetic mean, calculating factorial
  - The `break` and `continue` statements
    - `break`, `continue` within `while`, `do-while`
    - `break`, `continue` within `for`
    - Programming examples, e.g., testing whether a natural number is a prime or not, generating a sequence of integers skipping some terms
  - The `switch` statement
  - The `goto` statement

Unit 4. Functions and Arrays [11 Lecture Hrs]

- Functions
  - Overview of functions
    - General idea of a function
    - Benefits of using a function
  - Defining a function
    - General form
    - Arguments, parameters
    - Return types
  - Library functions
    - Header files
    - Functions from `math.h`
  - User-defined functions
    - Function prototypes
    - Examples of functions with different data types
    - Accessing a function
    - Local and global variables: scope rules

    - Passing arguments to a function
    - Programming examples, e.g., writing some of the previously discussed examples in function form, composition gof of functions f, g
  - Recursion
    - Recursion formula and recursive functions
    - Programming examples using recursions, e.g., to find $n!, \sum n$
- Arrays
  - Defining an array
    - Array declaration
    - Accessing elements of an array
    - Array initialization
  - Processing an array
    - Programming examples illustrating multiplication of a numerical data type array by a given number, storing the sum of individual entries in two arrays to a third array
    - Programming examples to find the average deviation, standard deviation of list of given numbers
  - Multi-dimensional array
    - One dimensional array vs. multidimensional array
    - Multidimensional array declaration
    - Accessing elements of a multidimensional array
    - Multidimensional array initialization
  - Arrays and strings
    - String as an array of characters
    - Initializing a string
    - String input and output

Unit 5. Solving Mathematical Problems with C Programming [21 Lecture Hrs]

- Searching and sorting
  - Searching for a particular element in a set using linear search
  - Searching for the maximum/minimum element in a set of real numbers
  - Sorting a given set of real numbers in ascending/descending order using bubble sort algorithm
- Matrices and matrix operations
  - Matrix as a multi-dimensional array
  - Matrix operations
    - Multiplication by a scalar
    - Transpose, testing whether a matrix is symmetric, skew-symmetric
    - Sum and difference
    - Matrix multiplication
  - Determinant of square matrices (up to 3×3 matrices)
- Numerical solution of a system of linear equations
  - Gauss-Seidel Method (diagonally dominant cases)
  - Use of Cramer's rule (no. of variables up to 3)
- Numerical solution of non-linear equations
  - Newton-Raphson method
- Solution of polynomial equations
  - Solution of quadratic equations
  - Identification of integer solutions of a polynomial equation
- Sequence and series
  - Generating a sequence with a given general term
  - Finding sum of a finite number of terms in a series
- Approximating limits and derivatives

  - Using intuitive ideas of limits and derivatives to approximate the limit, derivative of a function at a given point
- Approximation by Taylor series expansion
  - Generating Taylor series expansion of a function with known n-th derivative
  - Finding approximate value of a function at a given point using Taylor series expansion about the origin
- Numerical integration
  - Approximation of definite integrals using Trapezoidal rule

Part B: Laboratory Works **[75 Lecture Hrs]**

Unit 1: [12 Lecture Hrs]

1. Familiarization with a computer system, activities like creation of a folder, copying, pasting, deleting
2. Installation of C program IDE, creating a new .c file, saving in a folder
3. First C programs with printf, single line and multi-line comments
4. Declaration of variables with different data types, displaying output with different data types along with `\n`, `\t`
5. Error handling tricks

Unit 2: [12 Lecture Hrs]

1. Input/output with `gechar`, `putchar`, `gets`, `puts`
2. Input/output with `scanf`, `printf` and programs with arithmetic operators taking input with `scanf` and output with `printf`.
3. Programs for using simple mathematical formulae - calculation of simple interest, area of a circle, volume of a sphere, degree to Fahrenheit conversion, taking input
4. Programs to illustrate the use of the operators: `++`, `--`, `+=`, `-=`, `*=`, `/=`, `%=`, `?:`
5. Use of type conversion including binary to octal, hexadecimal, and vice versa
6. Programs to illustrate bitwise operators

Unit 3: [18 Lecture Hrs]

1. `if`, `if-else` statements, e.g., check whether a number is divisible by a given number or not, finding absolute value of a number
2. Nested `if-else` statements, e.g., assign the grade based on percentage of marks
3. `while` statements: e.g., create a list of numbers, find the average of numbers, find the factorial
4. `do-while` statements: e.g., take inputs of numbers unless 0 is entered, and find the sum of entered numbers
5. `for` looping: e.g., create a multiplication table from 1 to 10
6. `break`, `continue`, `goto`: e.g., creating a sequence until some criteria is met, skipping some terms of the sequence
7. `switch` statements: e.g., get two numbers and an operator symbol (+, -, *, /) as an input and display the sum, difference, product, quotient of the numbers accordingly
8. Problems with mixture of different control statements; e.g., testing whether a number is prime or not. Printing the list of prime numbers up to a given number

Unit 4: [15 Lecture Hrs]

1. Use of the functions using `#include<math.h>`
2. Examples of user-defined functions

- with and without arguments
- with and without return types

3. Construct two functions, and
   - define a third function which is a composite of the two functions and print its value,
   - define a function using other operations, e.g., sum, difference, quotient
4. Using recursion
   - find the factorial of a number,
   - find sum of the first n natural numbers.
5. Store a list of numbers in an array
   - Multiply each element in an array by a given number and store in a different array. Print the resulting array.
   - Store the sum of two arrays in a third array and print it.
6. - Find different descriptive statistics, e.g., mean, average deviation, standard deviation of the numbers stored in an array.
   - Store a matrix in a two-dimensional array. Access different elements of the matrix.
7. Programming examples involving
   - string as an array of characters
   - initializing a string
   - string input and output

Unit 5: [18 Lecture Hrs]

1. Represent a list of numbers as an array.
   - Test whether a number exists in the array.
   - Find the maximum or minimum of the elements of the array.
   - Sort the array in ascending/descending order using bubble sort algorithm.
2. Represent a matrix A as a two-dimensional array, and
   - construct a matrix C multiplying A by a constant, and print C,
   - construct D as a sum/difference of A and B, and print D,
   - take D as a matrix so that AD is defined. Define $E = AD$ and print E.
3. Consider a 3×3 matrix A as a two-dimensional array, and
   - test whether A is symmetric/skew-symmetric or not,
   - create a function to find the determinant of A.
4. Take a diagonally dominant system of linear equations with 2 or 3 variables.
   - Solve the system using Gauss-Seidel method.
   - Solve the system using Cramer's rule.
5. Consider a non-linear equation and an interval in which a solution lies. Find the solution using Newton-Raphson method.
6. - Given a quadratic equation, find its solution by using formula.
   - Given a cubic equation, test whether it has an integer root or not.
7. Given an n-th term of a sequence,
   - find the number of terms less than a given number,
   - print the sum of the first k terms for a given k.

8. Given $f(x)$,
   - approximate $f'(a)$ using the definition of derivative for a given a,
   - using the n-th derivative, generate the coefficients of $(x-a)^0, (x-a)^1, (x-a)^2, \cdots$ in the Taylor series expansion about a.
9. - Find the approximate value of a function at a point using Taylor series, e.g., $\log(1.1)$, $\sin 0.2$, e^2.
   - Use trapezoidal rule to find the definite integral of a given function, e.g., $\int_1^4 \log x \; dx, \int_0^1 e^{x^2} \; dx$. Find the error of approximation when the corresponding indefinite integral is known.

Text/Reference Books:

1. Y. P. Kanetkar: *Let Us C*. BPB publications, 2004.
2. B. S. Gottefried: *Theory and Problems of Programming with C*, Mc-Graw Hill, New York, 1996
3. E. Balagurusamy: *Programming in ANSI C (4E)*, Tata Mc Graw Hill, 1989.

Examination:

There will be a final examination of 70 marks (Theory: 35 marks, Practical: 35 marks) and internal examination(s) of 30 marks. The candidate must pass in theory part and practical (laboratory) part of the final examination respectively and internal (Theory and Practical) examinations separately.

The duration of the final examination of the theory part (35 marks) will be of one hour and 30 minutes. The examination for the practical (laboratory) part of 35 marks will be conducted by the concerned department of mathematics along with external examiner(s) and the marks will be submitted to Tribhuvan University Office of the Controller of Examination, Balkhu.

Marks allocation for the final practical examination:

- Laboratory examination: 20 marks
- Quiz (objective type questions): 5 marks
- Lab record file: 5 marks
- Viva-voce: 5 marks

The internal examination of 30 marks will be conducted by the department of mathematics of the related campus and the marks will be submitted to Tribhuvan University Office of the Controller of Examination, Balkhu.

Marks allocation for the internal examination:

- Theory examinations: 10 marks
- Laboratory examinations: 10 marks
- Presentations: 3 marks
- Assignments: 2 marks
- Quiz (objective questions): 2 marks
- Viva-voce: 3 marks

There will be an internal examination (Theory written exam and Quiz) of 12 marks for the period of one hour.

Guidelines to the question setter (Theory):
In the final examination
1. Questions must include every unit.
2. There will be two groups, namely, Group A and Group B.
3. In group A, there must be OR selection for 5 marks questions.
4. In group B , there must be OR selection for 10 marks questions.
5. OR Selection must be given from the same unit.
6. Questions must be creative and should be appropriate to the allocated time.

Guidelines to the question setter (Practical):
In the final examination
1.There is Quiz (objective type questions): 5 marks for the examination period of 15 minutes.
2. There must be OR selection for 10 marks questions of Laboratory examination (20 marks) for the examination period of 1 hour 15 minutes.
On the basis of the guidelines mentioned, we enclose one set of model question for Computer Programming (Math 410B)(Theory and Practical both)

Model Question

Tribhuvan University
Faculty of Humanities and Social Sciences

Bachelor Level/ III year/Humanities — Full Marks: 35
Mathematics (Math 410B) — Pass Marks: 14
Computer Programming(Theory) — Time: $1\frac{1}{2}$ hours

Candidates are required to give their answers in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt **All** the questions.

Group A **[1 × 15 = 15]**

1. (a) Mention, with examples, four basic data types used in C. Write a program to read values into any two floating point variables. Compute their sum and print it. The value of sum should be printed up to two decimal places only. Label your program with proper comments. [1 + 1 + 1 + 1 + 1]

OR

Mention any three rules of naming identifiers. Write a program to declare three variables of data type integer, double, and string. Read the values from user and print the values. Label your program with appropriate comments. [1 + 2 + 1 + 1]

(b) What is the purpose of if-else statement? Write a program to read marks of 4 subjects of a student. Compute percentage and display the grade using following rules: [1 + 4]

Percentage	Grade
Greater than or equal to 85	A
Less than 85 and greater than or equal to 80	B
Less than 80 and greater than or equal to 75	C
Less than 75	F

(c) What is the difference between the local variables and global variables? Write a C- program to find the factorial of a number using recursive method. [5]

Group B **[2 × 10 = 20]**

2. What are unary operators? Discuss the types of unary operators with examples. Write a C-program using do while loop to illustrate the use of increment operator. [2 + 3 + 5]

3. Write a program to approximate $\int_1^4 \log_e x\, dx$ using Trapezoidal rule. Using

$$\int \log_e x\, dx = x \log_e x - x,$$

find the error in the approximation. [7 + 3]

OR

Write a program to find the roots of a quadratic equation and display

(i) both roots if they are unequal,

(ii) only one root if they are equal. [5+5]

Model Question

Tribhuvan University
Faculty of Humanities and Social Sciences

Bachelor Level/ III year/Humanities — Full Marks: 35
Mathematics (Math 410B) — Pass Marks: 14
Computer Programming(Practical) — Time: $1\frac{1}{2}$ hours

A. Quiz (objective type questions): 15 minutes [5 marks]

1. Which of following is a valid C-identifier?
   a. 5th
   b. "c_identifier"
   c. **_suma**
   d. c-identifier

2. Which of following is a multiline comment in C?
   a. // hello
   b. **/*hello*/**
   c. //hello//
   d. /*hello%

3. Which of following is used in switch statement to exit out of a particular case?
   a. **break**
   b. exit
   c. case
   d. switch

4. What will be the output of following code?
```
#include<stdio.h>
void main()
{
int i = 0;
while (i < 3)
{
printf("Hello");
i++;
}

}
```
   a. `Hello` is printed zero times.
   b. `Hello` is printed twice.
   c. **`Hello` is printed thrice**.
   d. `Hello` is printed infinitely.

5. A program execution in C always begins by carrying out the instructions in
   a. **main function**
   b. printf function
   c. scanf function
   d. All of the options

6. Strings in C are terminated by
   a. **\0**
   b. \a
   c. \n
   d. \t

7. Function with no return type is represented by
   a. int
   b. float
   c. **void**
   d. double

8. A function to calculate natural logarithm with the header file "math.h" is:
   a) **log()**
   b) ln()
   c) loge()
   d) log10()

9. The function pow() to calculate power of a number belongs to the header file … …
   a) mathio.h
   b) square.h
   c) power.h
   d) **math.h**

10. In an array initialized as
    `int arr[3][2] = {{10, 20}, {30, 40}, {50, 60}};`
    the value of `arr[2][1]` is:
    a) **60**
    b) 30
    c) 20
    d) 21

B. Laboratory examination: 1 hr 15 minutes [$2 \times 10 = 20$ marks]

1. (a) Write a recursive program to find sum of first n natural numbers. [5]

   (b) Read a character type input and use switch statement to display "Male" if input is m, "Female" if input is f, "None" for default case. [5]

2. Write a program to multiply two matrices of sizes 2×3 and 3×2 and display the result. [10]

OR

Write a program for Newton-Raphson method to find an approximate root of non-linear equation $3x \sin x + e^x = 0$ with tolerance error 0.0001 and initial guess $a_0 = 1$. [10]

C. Lab record file [5 marks]
D. Viva-voce [5 marks]

8 Math 426 Detailed Course of Modern Algebra

TRIBHUVAN UNIVERSITY
Faculty of Humanities and Social Sciences
Micro Syllabus

Course Title: Modern Algebra (Compulsory)	**Full Marks:** 100
Course No.: Math 426	**Pass Mark:** 40
Level: B.A.	**Year:** IV
Nature of Course: Theory	**Periods:** 9 Lectures Hrs/Week

Course Objectives: This course is designed for the fourth year of Four years B.A. program as a compulsory subject in mathematics. The main objectives of this course are to enable the students to develop in-depth knowledge and good theoretical background in algebra, to take up higher studies to sustain interest and enjoyment of algebra and its applications in various branches of mathematics related to humanities and social sciences, and to get associated with teaching and familiar with recent trends in the field of algebra.

Detailed Course Contents:

Unit 1. Groups and Subgroups [13 Lecture Hrs]

Introduction and examples, Complex numbers, Euler's formula, Algebra on circles,
Binary operations: definition and examples,
isomorphism binary structures:
Groups: definition and examples, Elementary properties of groups
Theorem: If G is a group with binary operation *, then the left and right cancellation laws hold in G.
Theorem: If G is a group with binary operation *, and if a and b are any elements of G, then the linear equations a*x= b and y*a= b have unique solutions x and y in G.
Theorem: The identity element and inverse of each element are unique in a group.
Corollary: Let G be a group. For all $a, b \in G$, we have $(a * b)^{'} = b^{'} * a^{'}$.
Finite groups, order of a group, order of an element of a group and group tables,
Subgroups: notation and terminology, Subsets and subgroups, improper, proper, trivial and non trivial subgroups, examples,
Lemma:If H is any subgroup of a group G, then Hh= H=hH.
Theorem: A subset H of a group G is a subgroup of G if and only if
(a) H is closed under the binary operation of G,
(b) The identity element e of G is in H,
(c) For all $a \in H$ it is true that a^{-1} H also.
cyclic Group and subgroups,
Theorem: Let G be a group and let $a \in G$. Then $H = \{a^n : n \in \mathbb{Z}\}$ is a subgroup of G and every subgroup containing a contains H.
Examples,
Theorem: Every cyclic group is abelian.
Theorem: **Division Algorithm for** $\mathbb{Z}$. If m is a positive integer and n is any integer, then there exist unique integers q and r such that n= mq+r and $0 \leq r < m$.
Theorem: A subgroup of a cyclic group is cyclic.
Corollary: The subgroups of $\mathbb{Z}$ under addition are precisely the groups $n\mathbb{Z}$ under addition for $n \in \mathbb{Z}$.
Greatest common divisor
Theorem: Let G be a cyclic group with generator a. If the order of G is infinite, then G is isomorphic to ($\mathbb{Z}$, +). If G has finite order n, then G $(\mathbb{Z}_n, +_n)$, Generating set and Caley diagraphs,
Theorem: The intersection of some subgroups H_i of a group G for $i \in I$ is again a subgroupG.
Theorem (No Proof): If G is a group and $a_i \in G$ for i I, then the subgroup H of G generated by $\{a_i : i \in I\}$ has an element precisely those elements of G that are finite products of integral powers of the a_i, where powers of a fixed a_i may occur several times in the product.
Discussion about Cayley diagraph with examples,
Related problems.

Unit 2. Permutations, Cosets, and Direct Products [18 Lecture Hrs]

Groups of permutations, Permutation groups,
Theorem: Let A be a nonempty set, and let S_A be the collection of all permutations of A. Then S_A is a group under permutation multiplication.
Two important examples, Image of H,
Lemma: Let G and $G^{'}$ and let $\phi : G \to G^{'}$ be a one-to-one function such that $\phi(xy) = \phi(x)\phi(y)$ for all x, y

G. Then $\phi[G]$ is a subgroup of $G^{'}$ and ϕ provides an isomorphism of G with $\phi[G]$.
Theorem: **(Cayley's)**: Every group is isomorphic to a group of permutations.
Orbits, Cyclic, and the Alternating groups,
Theorem: Every permutation of a finite set is a product of disjoint cycles.
Even and odd permutation,
Corollary: Any permutation of a finite set of at least two elements is a product of transpositions.
Theorem (No Proof): No permutation in S_n can be expressed both as a product of even number of transposition and as a product of an odd number of transpositions.
Cosets and the theorem of Lagrange, Cosets with examples,
Theorem: Let H be a subgroup of G. Let the relation $\sim_L$ be defined on G by $a \sim_L b$ if and only if $a^{-1}b \in H$. Let $\sim_R$ be defined by $a \sim_R b$ if and only if $ab^{-1} \in H$. Then and $\sim_L$ and $\sim_R$ are both equivalence relations on G.
Theorem: Any two right(left) cosets of a subgroup are either disjoint or identical.
Theorem: (Theorem of Lagrange) Let H be a subgroup of a finite group G. Then the order of H is a divisor of the order of G.
Corollary: Every group of prime order is cyclic.
Theorem: The order of an element of a finite group divides the order of the group.
Direct product with examples,
Theorem: Let $G_1, G_2, , G_n$ be groups. For $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ in $\prod_{i=1}^{n} G_i$,
define $(a_1, a_2, ..., a_n)(b_1, b_2, ..., b_n)$ to be the element $(a_1b_1, a_2b_2, ..., a_nb_n)$. Then $\prod_{i=1}^{n} G_i$ is a group under this binary operation.
Internal direct product
Lemma: Suppose that G is the internal direct product of $N_1, N_2, ..., N_n$. Then for $i \neq j, N_i \cap N_j = \{e\}$,and if $a \in N_i, b \in N_j$ then ab=ba.
Theorem: Let G be a group and suppose that G is the internal direct product of $N_1N_2, ..., N_n$. Let $T = N_1 \times N_2 \times ... \times N_n$. Then G and T are isomorphic.
Theorem (No Proof): The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime, that is, the gcd of m and n is 1.
Corollary (No Proof): The group $\prod_{i=1}^{n} \mathbb{Z}_m$ is cyclic and isomorphic to $\mathbb{Z}_{m_1m_2...m_n}$ if and only if the numbers m_i for i= 1, 2, . . ., n are such that the gcd of any two of them is 1.
Theorem: Let $(a_1, a_2, ..., a_n) \in \prod_{i=1}^{n} G_i$. If a_i is of finite order r_i in G_i, then the order of $(a_1, a_2, ..., a_n)$ in $\prod_{i=1}^{n} G_i$ is equal to the least common multiple of all the r_i.
Related problems.

Unit 3. Homomorphism and Factor Groups [18 Lecture Hrs]

Homomorphism, Examples, Properties of homomorphism.
Theorem: Let ϕ be a homomorphism of a group G into a group $G^{'}$
(a) If e is the identity element in G, then $\phi(e)$ is the identity element $e^{'}$ in $G^{'}$,
(b) If $a \in G$, then $\phi(a^{-1}) = \phi(a)^{-1}$,
(c) If H is a subgroup of G, then $\phi[H]$ is a subgroup of $G^{'}$,
(d) If $K^{'}$ is a subgroup of $G^{'}$, then $\phi^{-1}[K^{'}]$ is a subgroup of G.
Theorem: Let $\phi : G \to G^{'}$ be a group homomorphism, and let $H = Ker(\phi)$. Let $a \in G$. Then the set $\phi^{-1}[\{\phi(a)\}] = x \in G : \phi(x) = \phi(a)$ is the left coset aH of H, and is also the right coset Ha of H. Consequently, the two partition of G into left cosets and into right cosets of H are the same.
Corollary: A group homomorphism $\phi : G \to G^{'}$ is a one to one map if and only if $Ker(\phi) = \{e\}$.
Normal subgroup with examples
Corollary: If $\phi : G \to G^{'}$ is a group homomorphism, then $Ker(\phi)$ is a normal subgroup of G.
Factor groups with examples
Theorem: Let $\phi : G \to G^{'}$ be a group homomorphism with kernel H. Then the cosets of H form a factor group,$\frac{G}{H}$, where (aH) (bH)= (ab) H. Also, the map $\mu : \frac{G}{H} \to \phi[G]$ defined by $\mu(aH) = \phi(a)$ is an isomorphism. Both coset multiplication and μ are well defined, independent of the choices a and b from the cosets.
Theorem: Let H be a subgroup of a group G. Then left coset multiplication is well defined by the equation (aH) (bH)= (ab) H if and only if H is a normal subgroup of G.
Corollary: Let H be a normal subgroup of G. Then the cosets of H form a group $\frac{G}{H}$ under the binary operation (aH) (bH)= (ab) H
Theorem: Let H be a normal subgroup of G. Then $\gamma : G \to \frac{G}{H}$ given by $\gamma(x) = xH$ is a homomorphism with kernel H .
Theorem (The Fundamental Homomorphism Theorem): Let $\phi : G \to G^{'}$ be a group homomorphism with kernel H. Then $\phi[G]$ is a group, and $\mu : \frac{G}{H} \to \phi[G]$ given by $\mu(gH) = \phi(g)$ is an isomorphism. If $\gamma : G \to \frac{G}{H}$ is the homomorphism given by $\gamma(g) = gH$, then $\phi(g) = \mu\gamma(g)$ for each $g \in G$.
Normal subgroups and inner automorphism

Theorem: The following are three equivalent conditions for a subgroup H of a group G to be a normal subgroup of G
(a) $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$
(b) $gHg^{-1} = H$ for all $g \in G$
(c) gH= Hg for all $g \in G$
also, find quotient groups,
Factor group computations and simple groups with examples, definition of simple group and examples
Theorem: Let $\phi : G \to G^{'}$ be a group homomorphism. If N is a normal subgroup G, then $\phi[N]$ is a normal subgroup of $\phi[G]$. Also, if $N^{'}$ is a normal subgroup of $\phi[G]$, then $\phi^{-1}[N^{'}]$ is a normal subgroup of G.
Related problems.

Unit 4. Group Theory Continued [13 Lecture Hrs]

Normalizer of a subgroup of a group, index of a subgroup H in a group G, Center of a group with examples
Theorem: The normalizer N(a) of any element a of a group G is a subgroup of G.
Theorem: The center Z(G) of a group G is a normal subgroup of G.
Index of H in G
Theorem: Suppose H and K are subgroup of a group G such that $K \leq H \leq G$, and suppose index (H:K) and (G:H) are both finite. Then (G:K) is finite, and (G:K)= (G:H)(H:K).
Isomorphism Theorems,
Theorem (First Isomorphism Theorem): Let $\phi : G \to G^{'}$ be a homomorphism with kernel K, and let $\gamma_K : G \to \frac{G}{K}$ be the canonical homomorphism. Then there is a unique isomorphism $\mu : \frac{G}{K} \to \phi[G]$ such that $\phi(x) = \mu(\gamma_K(x))$ for each $x \in G$.
Lemma: Let N be a normal subgroup of a group G and let $\gamma : G \to \frac{G}{N}$ be the canonical homomorphism. Then the map ϕ from the set of normal subgroups of G containing N to be the set of normal subgroups of $\frac{G}{N}$ given by $\phi(L) = \gamma[L]$ is one to one and onto.
Join of H and N, $H \vee N$
Lemma: If N is a normal subgroup of G, and if H is any subgroup of G, then $H \vee N$= HN= NH. Furthermore, if H is also normal in G, then HN is normal in G.
Theorem (Second Isomorphism Theorem): Let H be a subgroup of G and let N be a normal subgroup of G. Then $\frac{HN}{N} \cong \frac{H}{(H \cap N)}$.
Example
Theorem (Third Isomorphism Theorem): Let H and K be normal subgroups of a group G with K $\leq$ H. Then $\frac{G}{H} \cong \frac{\frac{G}{K}}{\frac{H}{K}}$.
Theorem: For a prime number p, every group G of order p^2 is abelian.
Theorem: If H and K are finite subgroups of a group G, then $|HK| = \frac{(|H|)(|K|)}{|H \cap K|}$.
Related problems.

Unit 5. Rings and Fields [13 Lecture Hrs]

Rings and fields, definition and basic properties, and examples
Theorem: If R is a ring with additive identity 0, then for $a, b \in R$ we have
(a) 0a= a0= 0,
(b) a(-b)= (-a)b= -(ab),
(c) (-a)(-b)= ab
Homomorphism and isomorphism with examples, Commutative ring, unit, division ring, skew field, subring, Divisors of zero, Integral domains, Examples,
Theorem: In the ring $\mathbb{Z}_n$, the divisors of 0 are precisely those nonzero elements that are not relatively prime to n.
Corollary: If p is a prime, then $\mathbb{Z}_p$ has no divisors of 0.
Theorem: The cancellation laws hold in a ring R if and only if R has no divisors of 0.
Theorem: Every field F is an integral domain.
Theorem: Every finite integral domain is a field.
Theorem: If p is a prime, then $\mathbb{Z}_p$ is a field.
The characteristic of a ring with examples
Theorem: Let R be a ring with unity. If $n.1 \neq 0$ for all $n \in \mathbb{Z}^+$, then R has characteristic 0. If n.1= 0 for some $n \in \mathbb{Z}^+$, then the smallest such integer n is the characteristic of R.
Fermat's and Euler's theorem
Theorem (Little Theorem of Fermat): If $a \in \mathbb{Z}$ and p is a prime not dividing a, then p divides $a^{p-1} - 1$, that is, $a^{p-1} \equiv 1 (\text{mod} p)$ for $a \not\equiv 0 (\text{mod} p)$
Corollary: If $a \in \mathbb{Z}$, then $a^p \equiv a (\text{mod} p)$ for any prime p
Examples

Theorem (No Proof): The set G_n of nonzero elements of $\mathbb{Z}_n$ that are not 0 divisors form a group under multiplication modulo n.
Theorem (Euler's Theorem): If a is an integer relatively prime to n, then $a^{\phi(n)} - 1$ is divisible by n, that is, $a^{\phi(n)} \equiv 1(\text{mod} n)$.
Theorem: Let m be a positive integer and let $a \in \mathbb{Z}_m$ be relatively prime to m. For each $b \in \mathbb{Z}_m$, the equation ax= b has a unique solution in $\mathbb{Z}_m$.
The field of quotients of an integral domain,
Lemma: The relation between elements of the set $S = \{(a, b) : a, b \in \mathbb{Z}\}$ as (a, b) and (c, d) in S denoted by $(a, b) \sim (c, d)$, if and only if ad=bc is an equivalence relation.
Lemma (No Proof): For [(a,b)] and [(c,d)] in F, the equations [(a,b)] + [(c,d)]= [(ad+bc,bd)] and [(a,b)] [(c,d)] = [(ac, bd)] give well- defined operations of addition and multiplication on F.
Lemma: The map $i : D \to F$ given by $i(a) = [(a, 1)]$ is an isomorphism of D with a subring of F, where D is an integral domain. Here we can take $D = \mathbb{Z}$
Rings of polynomials, Definition
Theorem: The set R[x] of all polynomials in an indeterminate x with coefficients in a ring R is a ring under polynomial addition and multiplication. If R is commutative, then so is R[x], and if R has unity $1 \neq 0$, then 1 is also unity for R[x].
Theorem (No Proof): **(The evaluation Homomorphisms for Field Theory)** Let F be a subfield of a field E, let α be any element of E, and let x be an indeterminate. The map $\phi_\alpha : F[x] \to E$ defined by $\phi_\alpha(a_0 + a_1x + a_2x^2 + ... + a_nx^n) = a_0 + a_1\alpha + a_2\alpha^2 + ... + a_n\alpha^n$ for $(a_0 + a_1x + a_2x^2 + ... + a_nx^n) \in F[x]$ is a homomorphism of F[x] into E. Also, $\phi_\alpha(x) = \alpha$, and ϕ_α maps F isomorphically by the identity map; that is $\phi_\alpha(a) = a$ for $a \in F$. The homomorphism ϕ_α is evaluation at α.
Examples,
Factorization of polynomials over a field,
Theorem (Division Algorithm for F[x]): Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ and $g(x) = b_mx^m + b_{m-1}x^{m-1} + ... + b_1x + b_0$ be two elements of F[x], with a_n and b_m both nonzero elements of F and $m > 0$. Then there are unique polynomials q(x) and r(x) in F[x] such that f(x)= g(x)q(x)+ r(x), whether either r(x)= 0 or the degree of r(x) is less than the degree m of g(x) .
Examples,
Corollary (Factor Theorem): An element $a \in F$ is a zero of $f(x) \in F[x]$ if and only if x-a is a factor of f(x) in F[x].
Examples
Corollary (No Proof): A nonzero polynomial $f(x) \in F[x]$ of degree n can have at most n zeros in a field F.
Irreducible polynomials with examples,
Theorem: Let $f(x) \in F[x]$, and let f(x) be of degree 2 or 3. Then f(x) is reducible over F if and only if it has a zero in F.
Theorem (No Proof): **(Eisenstein Criterion)** Let $p \in \mathbb{Z}$ be a prime. Suppose that
$f(x) = a_nx^n + ... + a_1x + a_0$ is in $\mathbb{Z}[x]$, and $a_n \not\equiv 0(\text{mod} p)$, but $a_i = 0(\text{mod} p)$ for all $i < n$, with $a_0 \not\equiv 0(\text{mod } p^2)$. Then f(x) is irreducible over $\mathbb{Q}$.
In other words, let $p \in \mathbb{Z}$ be a prime. Suppose that $f(x) = a_nx^n + ... + a_1x + a_0$ is in $\mathbb{Z}[x]$.
If $p|a_0, p|a_1, p|a_2, ..., p|a_{n-1}, p \nmid a_n, p^2 \nmid a_0$, then f(x) is irreducible polynomial over $\mathbb{Q}$.
Examples,
Related problems.

Unit 6. Ideals and Factor Rings [18 Lecture Hrs]

Homomorphisms and factor rings with examples
Theorem: Let ϕ be a homomorphism of a ring R in into a ring R'. If 0 is the additive identity in R, then $\phi(0) = 0'$ is the additive identity in R', and if $a \in R$, then $\phi(-a) = -\phi(a)$. If S is a subring of R, then $\phi(S)$ is a subring of R'. Going the other way, if S' is a subring of R', then $\phi^{-1}[S']$ is a subring of R. Finally, if R has unity 1, then $\phi(1)$ is unity for $\phi[R]$.
Theorem: Let $\phi : R \to R'$ be a ring homomorphism, and let $H = \text{Ker}(\phi)$.Let $a \in R$. Then $\phi^{-1}[\phi(a)] = a + H = H = a$, where a+ H= H+ a is the coset containing a of the commutative additive group $< H, + >$.
Corollary: A ring homomorphism $\phi : R \to R'$ is a one-to-one map if and only if $\text{Ker}(\phi) = \{0\}$.
Theorem: Let $\phi : R \to R'$ be a ring homomorphism with kernel H. Then the additive cosets of H form a ring $\frac{R}{H}$ whose binary operations are defined by choosing representatives. That is, the sum of two cosets is defined by
(a+ H)+ (b+ H)= (a+ b)+ H,
and the product of the cosets is defined by
(a+ H)(b+ H)= (ab+ H).
Also, the map $\mu : \frac{R}{H} \to \phi[R]$ defined by $\mu(a + H) = \phi(a)$ is an isomorphism.

Theorem: Let H be a subring of the ring R. Multiplication of additive cosets of H is well defined by the equation
(a+ H) (b+ H)= ab + H
if and only if $ah \in H$ and $hb \in H$ for all $a, b \in R$ and $h \in H$.
Ideal with examples,
Corollary: Let N be an ideal of a ring R. Then the additive cosets of N form a ring $\frac{R}{N}$ with the binary operations defined by
(a+ N)+(b+ N)= (a+ b)+N
and
(a+ N)(b+ N)= ab+ N.
Factor ring
Theorem: Let N be an ideal of a ring R. Then $\gamma : R \rightarrow \frac{R}{N}$ given by $\gamma(x) = x + N$ is a ring homomorphism with kernel N.
Theorem (Fundamental homomorphism theorem): Let $\phi : R \rightarrow R^{'}$ be a ring homomorphism with kernel N. Then $\phi[R]$ is a ring, and the map $\mu : \frac{R}{N} \rightarrow \phi[R]$ given by $\mu(x + N) = \phi(x)$ is an isomorphism. If $\gamma : R \rightarrow \frac{R}{N}$ is the homomorphism given by $\gamma(x) = x + N$, then for each $x \in R$, we have $\phi(x) = \mu\gamma(x)$.
Prime and maximal ideals with examples
Theorem: If R is a ring with unity, and N is an ideal of R containing a unit, then N= R.
Corollary: A field contains no proper nontrivial ideals.
Theorem: Let R be a commutative ring with unity. Then M is a maximal ideal of R if and only if $\frac{R}{M}$ is a field.
Corollary: A commutative ring with unity is a field if and only if it has no proper nontrivial ideals.
Prime ideal with examples,
Theorem: Let R be a commutative ring with unity, and $N \neq R$ be an ideal in R. Then $\frac{R}{N}$ is an integral domain if and only if N is a prime ideal in R.
Corollary: Every maximal ideal in a commutative ring R with unity is a prime ideal.
Theorem: If R is a ring with unity 1, then the map $\phi : \mathbb{Z} \rightarrow R$ given by $\phi(n) = n.1$ for $n \in \mathbb{Z}$ is a homomorphism of $\mathbb{Z}$ into R.
Related problems.

Unit 7. Extension Fields [13 Lecture Hrs]

Definition of field, extension field with examples,
Theorem (No Proof) If F is a field, then every nonconstant polynomial $f(x) \in F[x]$ can be factored in F[x] into a product of irreducible polynomial, the irreducible polynomials being unique except for order and for unit (that is, nonzero constant) factors in F.
Theorem (Kronecker's theorem): (Basic goal) Let F be a field and let f(x) be a nonconstant polynomial in F[x].Then there exists an extension field E of F and $\alpha \in E$ such that $f(\alpha) = 0$
Algebraic and transcendental elements with examples,
Theorem: Let E be an extension field of a field F and let $\alpha \in E$. Let $\phi_\alpha : F[x] \rightarrow E$ be the evaluation homomorphism of F[x] into E such that $\phi_\alpha(a) = a$ for $a \in$ F and $\phi_\alpha(x) = \alpha$. Then α is transcendental over F if and only if ϕ_α gives an isomorphism of F[x] with a subdomain of E, that is, if and only if ϕ_α is a one-to-one map.
The irreducible polynomial for over F,
Theorem: Let E be an extension field of a field F, and let $\alpha \in E$, where α is an algebraic over F. Then there is an irreducible polynomial $p(x) \in F[x]$ such that $p(\alpha) = 0$. This irreducible polynomial p(x) is uniquely determined up to a constant factor in F and is a polynomial of minimal degree ≥ 1 in F[x] having α as a zero. If $f(\alpha) = 0$ for $f(x) \in F[x]$, with $f(x) \neq 0$, then p(x) divides f(x).
The monic polynomial with examples, simple extension with example,
Theorem: Let E be a simple extension of $F(\alpha)$ of a field F, and let α be algebraic over F. Let the degree of irr(α, F) be $n \geq 1$. Then every element β of $E = F(\alpha)$ can be uniquely expressed in the form

$$\beta = b_0 + b_1\alpha + b_2\alpha^2 + ... + b_{n-1}\alpha^{n-1}$$

, where the b_i are in F.
Algebraic extensions,
Theorem: A finite extension field E of a field F is an algebraic extension of F.
Theorem: If E is finite extension field of a field F, and K is a finite extension field of E,then K is a finite extension of F, and

$$[K : F] = [K : E][E : F]$$

.
Corollary (No Proof): If E is an extension field of F, αE is algebraic over F, and $\beta \in F(\alpha)$, then deg(β, F)

divides $\deg(\alpha, F)$.
Theorem: Let E be an algebraic extension of a field F. Then there exist a finite number of elements $\alpha_1, \alpha_2, ..., \alpha_n$ in E such that $E = F(\alpha_1, \alpha_2, ..., \alpha_n)$ if and only if E is a finite- dimensional vector space over F, that is, if and only if E is a finite extension of F.
Examples,
Related problems.

Unit 8. Factorization [13 Lecture Hrs]
Principal ideal, principal ideal domain generated by a, Examples,
Theorem: If F is a field, every ideal in F[x] is principal.
Theorem: An ideal $< p(x) > \neq 0$ of F[x] is maximal if and only if p(x) is irreducible over F.
Theorem: Let p(x) be an irreducible polynomial in F[x]. If p(x) divides r(x) s(x) for $r(x), s(x) \in F[x]$, then either p(x) divides r(x) or p(x) divides s(x).
Definition of principal ideal domain (PID) and unique factorization domain (UFD) with examples,
Theorem (No Proof): Every PID is a UFD.
Definition of primitive polynomial and content with examples,
Lemma (No Proof): If D is a UFD, then for every nonconstant $f(x) \in D[x]$ we have f(x)= (c) g(x), where $c \in D, g(x) \in D[x]$, and g(x) is primitive. The element c is unique up to a unit factor in D and is the content of f(x). Also, g(x) is unique up to a unit factor in D.
Lemma (Gauss's Lemma): If D is a UFD, then a product of two primitive polynomials in D[x] is again primitive.
Euclidean domains with examples,
Theorem: Every Euclidean domain is a PID.
Corollary: A Euclidean domain is a UFD.
Theorem: For a Euclidean domain with a Euclidean norm $\nu, \nu(1)$ is minimal among all $\nu(a)$ for nonzero $a \in D$, and $u \in D$ is a unit if and only if $\nu(u) = \nu(1)$.
Relation between Euclidean domain, UFD and PID,
Theorem (No Proof): (Euclidean Algorithm)
Examples,
Gaussian integers with examples
Lemma: In $\mathbb{Z}[i]$, the following properties of the norm function N hold for all$\alpha, \beta \in \mathbb{Z}[i]$:
(1) $N(\alpha) \geq 0$
(2) $N(\alpha) = 0$ if and only if $\alpha = 0$
(c) $N(\alpha\beta) = N(\alpha)N(\beta)$.
Lemma: $\mathbb{Z}[i]$ is an integral domain.
Related problems.

Unit 9. Theory of Polynomial Equations [18 Lecture Hrs]
Polynomial over an integral domain, Division algorithm.
Theorem: The set D[x] of all polynomials in x over an integral domain D, is an integral domain.
Theorem: For any two nonzero polynomials f(x) and g(x) over a field K, there exist unique polynomials q(x) and r(x) such that f(x) = g(x)q(x)+ r(x) where r(x) is zero or of degree less than that of g(x).
Theorem: (Remainder Theorem) The value of a polynomial(x) at x= c is equal to the remainder obtained on dividing f(x) by x- c.
Theorem: (Factor Theorem) x- c is a factor of a polynomial f(x) if and only if f(x) is zero.
Division of a polynomial, Zero of a polynomial, Rolle's Theorem (no proof),Properties of equations, Properties of equations, Descartes rule of signs.
Theorem: α is a root of the equation f(x) = 0 if and only if the polynomial f(x) is divisible by x- α.
Theorem: A polynomial over $\mathbb{C}$, of degree n, has exactly n zeros in $\mathbb{C}$.
Theorem: In an equation f(x) = 0 with real coefficients, imaginary roots occur in conjugate pairs.
Related problems discussion.
Relation between roots and coefficients, Application to the solution of an equation with related problems discussion.
Symmetric function of roots, Transformation of equations with related problems discussion.
Transformation in general, multiple roots, Sum of the power of roots with related problems discussion.
Reciprocal equations, Binomial equation with related problems discussion.Polynomial over an integral domain, Division algorithm, Division of a polynomial, Zeros of a polynomial, Rolle's theorem (no proof), Properties of equations, Descartes rule of signs, Relation between roots and coefficients, Application to the solution of an equation, Symmetric function of roots, Transformation of equations,Transformation in general, Multiple roots, Sum of the power of roots, Reciprocal equations, Binomial equation.
Related problems.

Unit 10. Cubic and Biquadratic Equations [13 Lecture Hrs]

Algebraic solution, Algebraic solution of the cubic, Nature of roots of cubic.
Equation of square difference of cubic, Nature of roots from Cardan's solution and application to the numerical examples with related problems discussion.
Cardan method with related problems discussion.
Solution by symmetric functions of roots, Solution of the biquadratic and the radical.
Related problems discussion about Ferrari's method.
Related problems discussion about radical (Euler's method).
Related problems discussion about Descarte's method.
Algebraic solution, Algebraic solution of the cubic, Nature of roots of cubic, Equation of square difference of cubic, Nature of roots from Cardan's solution and application to the numerical examples, Solution by symmetric functions of roots, Solution of the biquadratic and the radical.
Related problems.

Text/Reference Books:

1. J.B. Fraleigh: *A First Course in Abstract Algebra*, Seventh Edition, Pearson.
2. T.P. Nepal and T.B. Budhathoki: *A Textbook of Modern Algebra with Theory of Equations*, Heritage Publication and Distributors Pvt. Ltd. Kathmandu.
3. R.M. Shrestha and S. Bajracharya: *Linear Algebra, Groups, Rings & Theory of Equations*, Sukunda Pustak Bhavan, Kathmandu.
4. I.N. Herstein: *Topics in Algebra*, Vikas Publication, India.
5. A.R. Vasishtha; *Modern Algebra*, Krishna Prakashan Mandir, Meerut.

Examination:
There will be a final examination of 70 marks for the period of three hours. The internal examination of 30 marks will be conducted by the department of mathematics of related campus and the marks will be submitted to Tribhuvan University Office of the Controller of Examination, Balkhu. A candidate must pass the internal and the final examinations separately.
Marks allocation for the internal examination:

- Written examinations: 20 marks (1 hour)
- A student or a group of students with presentation: 5 marks
- Assignments: 5 marks

Guidelines to the question setter:
In the final examination
1. Questions must include every unit.
2. There will be two groups, namely, Group A and Group B.
3. In group A, there must be OR selection for 15 marks questions.
4. In group B , there must be OR selection for 20 marks questions.
5. OR Selection must be given from the same unit.
6. Questions must be creative and should be appropriate to the allocated time.

On the basis of the guidelines mentioned, we enclose one set of model question for Modern Algebra (Math 426)

Model Question

Tribhuvan University
Faculty of Humanities and Social Sciences

Bachelor Level/ IV year/Humanities — Full Marks: 70
Mathematics (Math 426) — Pass Marks: 28
Modern Algebra — Time: 3 hours

Candidates are required to give their answers in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt **All** the questions.

Group A $[2 \times 15 = 30]$

1. (a) Define permutation group with an example. Prove that every group is isomorphic to a group of permutations. Verify this theorem to take a group $G = \{1, \omega, \omega^2\}$, where $\omega^3 = 1$, is isomorphic to a subgroup of a permutation group of order 6. [2+4+1.5]

OR

Define direct product of the groups. Prove that the group $\mathbb{Z}_2 \times \mathbb{Z}_3$ is a cyclic group. Also, suppose that G is the internal direct product of $N_1, N_2, ..., N_n$. Then prove that for $i \neq j, N_i \cap N_j = \{e\}$, and if $a \in N_i, b \in N_j$ then ab=ba. [1+2+4.5]

(b) Define maximal ideal. Let R be a commutative ring with unity. Then prove that M is a maximal ideal of R if and only if $\dfrac{R}{M}$ is a field. Use this result to show that $3\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$ if and only if $\dfrac{\mathbb{Z}}{3\mathbb{Z}}$ is a field. [1+4.5+2]

2. (a) Define group. Let $G = \mathbb{R} - \{1\}$. Define * of G by a*b= a+b-ab. Then
(i) show that * gives a binary operation of G,
(ii) show that (G,*) is a group, (iii) find the solution of the equation 3*x*4=7 in G. [1+1.5+4+1]

(b) what do you mean by symmetric function? Find the sum of the squares and the cubes of the roots of the cubic equation $x^3 - px^2 + qx - r = 0$. Also, solve the equation $x^3 - 9x^2 + 14x + 24 = 0$ two of whose roots are in the ratio 3:2. [1.5+3+3]

OR

Find the condition that the equation $x^3 - px^2 + qx - r = 0$ should have its roots in harmonic progression and hence or otherwise solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$. [4.5+3]

Group B $[4 \times 10 = 40]$

3. (a) Define factor group. Find the factor group and the order of $\dfrac{\mathbb{Z}_6}{<2>}$ [5]

OR

Let H be a normal subgroup of agroup G. Prove that the following are equivalent:
(i) $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.
(ii) $gHg^{-1} = H$ for all $g \in G$.
(iii) gH=Hg for all $g \in G$. [5]

(b) Let $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_3$ be the homomorphism such that $\phi(1) = 2$. Then
(i) Find the kernel K of ϕ.
(ii) Find the group $\phi[\mathbb{Z}_{12}]$. [2+3]

4. Prove that if $a \in \mathbb{Z}$ and p is a prime not dividing a, the p divides a^{p-1}, that is, $a^{p-1} \equiv 1 (\bmod p)$ for $a \not\equiv 0 (\bmod p)$. How does Fermat's theorem become from Euler's theorem? Justify. Also, compute the remainder of 8^{103} when divided by 13 to use Fermat's theorem. [5+2+3]

OR

Prove that the set R[x] of all polynomials in an indeterminate x with coefficient in a ring R is a ring under polynomial addition and multiplication. Also, show that if R is a commutative ring, then so is R[x]. [8+2]

5. (a) Show that the number $\alpha = \sqrt{1 + \sqrt[3]{2}}$ is algebraic over $\mathbb{Q}$ by finding $f(x) \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$. [5]

(b) If D is a unique factorization domain, then prove that a product of two primitive polynomials in D[x] is again a primitive. [5]

OR

Define Euclidean domain with an example. Prove that every Euclidean domain is a principal ideal domain. [5]

6. Find the equation whose roots are the squares of the differences of the roots of the given cubic equation $x^3+qx+r = 0$. Hence form the equation of the squared differences of the cubic equation $x^3+6x^2+7x+2 = 0$.

[6+4]

9 Math 427 Detailed Course of Mathematical Analysis

TRIBHUVAN UNIVERSITY
Faculty of Humanities and Social Sciences
Micro Syllabus

Course Title: Mathematical Analysis (Compulsory) — **Full Marks:** 100
Course No.: Math 427 — **Pass Mark:** 40
Level: B.A. — **Year:** IV
Nature of Course: Theory — **Periods:** 9 Lectures Hrs/Week

Course Objectives: This course is designed for the fourth year of Four years B.A. program as a compulsory subject in mathematics. The main aim of this course is to provide advanced knowledge of mathematical analysis. Prerequisite for this course is the Real Analysis (Math 425). The general objectives of this course are:

a. to develop theoretical knowledge and analytical skills in the emerging areas of mathematics;

b. to raise interest of students in the field of analytical world so that they can take up any course easily in modern mathematics;

c. to acquire and develop skill in the use and understanding of mathematical language, especially in mathematical terms, symbol, statement,formula, definition and logic;

d. to construct solutions, proofs, examples and counter examples with their own independent efforts

e. to prepare a sound base for higher studies in pure and applied mathematics.

Detailed Course Contents:

UNIT 1 EUCLIDEAN SPACES AND METRIC SPACES [18 Lecture Hrs]

1.1 The Set $\mathbb{R}^n$ [2 Lectures]

1. Definitions of n-dimensional point and the set $\mathbb{R}^n$
2. Algebraic structure of $\mathbb{R}^n$: Definition of equality, sum, multiplication by scalars, difference, zero vector, inner product, norm and distance of n-dimensional points.
3. Vector space and Algebraic properties of $\mathbb{R}^n$.

Theorem (Statement only): $\mathbb{R}^n$ forms a vector space over the field of real numbers with respect to the coordinate wise addition and scalar multiplication.
Theorem (Statement only): For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^\mathbf{n}$ and any real number a

(i) $\mathbf{x}.\mathbf{y} = \mathbf{y}.\mathbf{x}$ (Commutative law)
(ii) $\mathbf{x}.(\mathbf{y}+\mathbf{z}) = \mathbf{x}.\mathbf{y} + \mathbf{x}.\mathbf{z}$ (Distributive law)
(iii) $(a\mathbf{x}).\mathbf{y} = a(\mathbf{x}.\mathbf{y}) = \mathbf{x}.(a\mathbf{y})$ (Associative law)
(iv) $\mathbf{x}.\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = \mathbf{0}$ and $\mathbf{x}.\mathbf{x} \neq \mathbf{0}$ for $\mathbf{x} \neq \mathbf{0}$

Theorem: Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\mathbf{n}$ and a be a real number. Then,

$$
\begin{array}{ll}
(i) & ||\mathbf{x}|| \geq \mathbf{0} \text{ and } ||\mathbf{x}|| = \mathbf{0} \text{ if, and only if, } \mathbf{x} = \mathbf{0} \\
(ii) & ||\mathbf{x}|| = ||-\mathbf{x}|| \\
(iii) & ||a\mathbf{x}|| = |a|\ ||\mathbf{x}|| \quad \text{(Absolute homogenity)} \\
(iv) & ||\mathbf{x}-\mathbf{y}|| = ||\mathbf{y}-\mathbf{x}|| \quad \text{(Symmetry)} \\
(v) & ||\mathbf{x}+\mathbf{y}||^\mathbf{2} + ||\mathbf{x}-\mathbf{y}||^\mathbf{2} = \mathbf{2}\left(||\mathbf{x}||^\mathbf{2} + ||\mathbf{y}||^\mathbf{2}\right) \quad \text{(Parallelogram equality)} \\
(vi) & \text{If } \mathbf{x}.\mathbf{y} = \mathbf{0}, \text{then } ||\mathbf{x}+\mathbf{y}||^\mathbf{2} = ||\mathbf{x}||^\mathbf{2} + ||\mathbf{y}||^\mathbf{2} \quad \text{(Pythagorean identity)}
\end{array}
$$

Theorem: Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\mathbf{n}$. Then,

$$
\begin{array}{ll}
(i) & |\mathbf{x}.\mathbf{y}| \leq ||\mathbf{x}||||\mathbf{y}|| \quad \text{(Cauchy-Schwarz's inequality)} \\
(ii) & ||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \quad \text{(Triangle inequality)} \\
(iii) & |||\mathbf{x}|| - ||\mathbf{y}||| \leq ||\mathbf{x}-\mathbf{y}||
\end{array}
$$

(d) Definition of unit coordinate vectors in $\mathbb{R}^n$.

1.2 Open Balls and Open Sets in $\mathbb{R}^n$ [3 Lectures]

1. Definitions of open n-balls, closed balls and spheres with examples and geometrical shape for different n.
2. Definitions of (i) interior point and interior (ii) exterior point and exterior (iii) boundary point and boundary of a set in $\mathbb{R}^n$.
3. Definition of open and closed sets in $\mathbb{R}^n$ with examples.
   Theorem: Let $S \subseteq \mathbb{R}^n$. Then, (i) S is open in $\mathbb{R}^n$ if, and only if, $S = \text{int}S$.
   (ii) The interior of S is an open subset of S.
4. Construction of open and closed sets in $\mathbb{R}^n$.
   Theorem: The union of arbitrary collection of open sets in $\mathbb{R}^n$ is open, and the intersection of a finite collection of open sets is open in $\mathbb{R}^n$.
   Theorem: The union of a finite collection of closed sets in $\mathbb{R}^n$ is closed, and the intersection of arbitrary collection of closed sets in $\mathbb{R}^n$ is closed in $\mathbb{R}^n$.
   Examples to show arbitrary intersection of open sets may not be open, and the arbitrary union of closed sets may not be closed.
   Theorem: IfA is closed and B is open in $\mathbb{R}^n$, then $A - B$ is closed and $B - A$ is open in $\mathbb{R}^n$.

1.3 Closed Sets and Adherent Points [3 Lectures]

1. Definition of adherent point and closure; accumulation point and derived set and isolated point of a set with examples.
   (a) **Theorem:** Let $S \subset \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$, then $\mathbf{x}$ is an accumulation point of S if, and only if, $\mathbf{x}$ is an adherent point of $S - \{\mathbf{x}\}$.
   (b) **Theorem:** $\mathbf{x}$ is an accumulation point of S if, and only if, every open n-ball $B(\mathbf{x})$ contains infinitely many points of S.
   (c) Examples of an infinite set with no accumulation point.
   (d) **Theorem:** A set $S \subseteq \mathbb{R}^n$ is closed if, and only if, it contains all its adherent points.
2. Definition of closure and derived set with examples
   (a) Proof of the relation $\overline{S} = S \cup S'$.
   (b) **Theorem:** A set S in $\mathbb{R}^n$ is closed if, and only if, S contains all its accumulation points.
   (c) **Theorem:** A set S in $\mathbb{R}^n$ is closed if, and only if, S contains all its boundary points.
   (d) **Theorem:** The derived set S' of a set S in $\mathbb{R}^n$ is a closed set.

1.4 Definition of Dense Set in $\mathbb{R}^n$ and Examples [0.5 Lectures]

1.5 Metric Spaces. [1.5 Lectures]

1. Definition of metric space with examples
2. Examples of metric spaces (Discrete metric space, $\mathbb{R}^n$ with different types of metrices)
3. Definition of metric subspace with examples

1.6 Point Set Topology in Metric Spaces [3 Lectures]

1. Definition of open ball in metric spaces and subspaces
2. The notation $B_S(a, r) = B_M(a, r) \cap S$
3. Definition of interior point, open set and closed set in metric space with examples
4. Construction of open and closed sets in metric space
5. Definition of adherent point, accumulation point, closure and derived sets in a metric space

Theorem: Let (S, d) be a metric subspace of a metric space (M, d), and X be a subset of S. Then, X is open in S if, and only if, $X = A \cap S$ for some set A which is open in M.
Theorem: Let (S, d) be a metric subspace of (M, d) and Y, a subset of S. Then Y is closed in S if, and only if, $X = B \cap S$ for some set B which is closed in M.
Theorem: The union of any collection of open sets is open, and intersection of a finite collection of open sets is open.
Theorem: The union of a finite collection of closed sets in a metric space is closed, and intersection of any collection of closed sets is closed.
Theorem: If A is open set in a metric space M and B is closed in M, then $A - B$ is open in M and $B - A$ is closed in M.
Theorem: For any subset S of a metric space M, the following statements are equivalent.

a) S is closed in M.
b) S contains all its adherent points.
c) S contains all its accumulation points.

d) $S = \overline{S}$.

1.7 Bolzano-Weierstrass Theorem [1 Lecture]

1. Definition of a bounded set in $\mathbb{R}^n$ and metric spaces.
2. Bolzano-Weierstrass theorem (No proof for $n > 1$).

1.8 Cantor's Intersection Theorem (Statement and Proof). [1 Lecture]

1.9 Exercise [3 Lectures]

At least the following problems from Exercise on page 65, Chapter-3 of the book Mathematical Analysis, Tom M. Apostol II Edition (1987) should be discussed:
$3.9, 3.11, 3.12, 3.26, 3.27, 3.28, 3.29, 3.31$

UNIT 2 COVERING THEOREMS AND COMPACTNESS [13 Lecture Hrs]

2.1 **Lindelöf Covering Theorem** [3 Lectures]

(a) Definition of covering, open covering, countable covering, finite covering and subcovering with examples.

(b) **Theorem:** Let $G = \{A_1, A_2, \cdots\}$ denote a countable collection of all n-balls having rational radii and centres at points with rational co-ordinates. Assume $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$ and let S be an open set in $\mathbb{R}^n$ such that $\mathbf{x} \in S$. Then, there exists at least one of the n-balls A_k in G contains x and is contained in S. That is, we have $\mathbf{x} \in A_k \subseteq S$ for some A_k in G.

(c) **Theorem:** (Lindelöf covering theorem in $\mathbb{R}^n$) (Statement and Proof)

2.2 **Heine-Borel Covering Theorem** [1 Lecture]

Theorem: Let F be an open covering of a closed and bounded set A in $\mathbb{R}^n$. Then there is a finite subcollection of F which also covers A.

2.3 **Compactness in $\mathbb{R}^n$** [2 Lectures]

(a) Definition of compact sets in $\mathbb{R}^n$ with examples.

(b) **Theorem:** Let S be a subset of $\mathbb{R}^n$. Then the following three statements are equivalent.
   i. S is compact.
   ii. S is closed and bounded.
   iii. Every infinite subset of S has an accumulation point in S.

2.4 **Compact Subsets of a Metric Space** [3.5 Lectures]

(a) Definition of open covering, compact set and bounded set in a metric space

(b) **Theorem:** Let S be a compact subset of a metric space M. Then,
   i. S is closed and bounded.
   ii. Every infinite subset of S has an accumulation point in S.

(c) Example to show that there exists a closed and bounded set in a metric space which is not compact.

(d) **Theorem:** Let X be a closed subset of a compact metric space M. Then X is compact.

(e) **Theorem:** Assume that $S \subseteq T \subseteq M$, (M, d) is a metric space. Then S is compact in (M, d) if, and only if, S is compact in the metric space (T, d).

2.5 **Exercise** [3.5hr]

At least the following problems from the Exercise on page 66 and 68, Chapter-3 of the book "Mathematical Analysis", Tom M Apostol II Edition (1987) should be discussed: $3.18, 3.19, 3.38, 3.39, 3.40, 3.41, 3.42$.

Unit 3 LIMITS AND CONTINUITY [18 Lecture Hrs]

3.1 **Convergent Sequences in Metric Spaces** [2 Lectures]

(a) Definition of convergent sequence with examples, definition of increasing, decreasing sequence, subsequence and bounded sequence

(b) The sequence $\left\{\dfrac{1}{n}\right\}$ converges in $\mathbb{R}$ but not in $(0, 1]$.

(c) **Theorem** (Uniqueness of limit): A sequence $\{x_n\}$ of points in a metric space (S, d) can converge to at most one point in S.

(d) **Theorem:** An increasing sequence $\{x_n\}$ which is bounded above converges to the supremum of its range $\{x_1, x_2, \cdots\}$ and a decreasing sequence $\{x_n\}$ which is bounded below converges to the infimum of its range $\{x_1, x_2, \cdots\}$.

(e) **Theorem:** Let $\{x_n\}$ be a sequence of points in a metric space (S, d) and let $x_n \to p$. If $T = \{x_1, x_2, \cdots\}$ is the range of $\{x_n\}$, then

i. T is bounded.

ii. p is an adherent point of T.

(f) **Theorem:** Let (S, d) be a metric space and $T \subseteq S$. Let p be a point in S which is an adherent point of T. Then, there is a sequence $\{x_n\}$ of points in T converging to p.

(g) **Theorem:** A sequence $\{x_n\}$ of points in a metric space (S, d) converges to p if, and only if, its every subsequence $\{x_{k(n)}\}$ also converges to p.

(h) **Theorem** (Sequential criterion for closed set): Let (S, d) be a metric space and A be a non empty subset of S. Then $p \in \overline{A}$ if, and only if, there exists a sequence $\{x_n\}$ in A such that $x_n \to p$.

3.2 **Cauchy Sequences** [1.5 Lectures]

(a) Definition of Cauchy sequence with examples

(b) **Theorem:** Cauchy's general principle of convergence or Cauchy convergence criterion

(c) **Theorem:** Every convergent sequence in a metric space is Cauchy but not conversely.

(d) **Theorem:** In Euclidean space $\mathbb{R}^k$, every Cauchy sequence is convergent.

3.3 **Complete Metric Space** [1 Lecture]

(a) Definition of complete metric space with examples.

(b) **Theorem:** In a metric space (S, d), every compact subset T is complete.

3.4 **Sequences and Compactness** [1 Lecture]

(a) Definition of sequentially compact metric space with example

(b) **Theorem:** If a metric space (S, d) is sequentially compact, then every infinite subset of S has a limit point.

(c) **Theorem:** Every compact metric space (S, d) is sequentially compact.

3.5 **Bolzano Weierstrass Theorem for Sequence** [0.5 Lectures]

Theorem (Bolzano-Weierstrass theorem for sequence): Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

3.6 **Limit of a Function and Properties** [1.5 Lectures]

(a) Definition of a limit of function in terms of ϵ, δ, and in terms of open balls

(b) **Theorem** (Sequential criterion for limit of a function): Let (S, d_S) and (T, d_T) be two metric spaces, $A \subseteq S$ and $f : A \to T$ function. Let p be an accumulation point of A and assume that $b \in T$. Then $\lim_{x\to p} f(x) = b$ if, and only if, $\lim_{n\to\infty} f(x_n) = b$, for every sequence $\{x_n\}$ of points of $A - \{p\}$ which converges to p.

(c) **Theorem** (Algebra of limit of functions): Let (S, d) be a metric space, $A \subseteq S$ and p be an accumulation point of A. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be two real valued functions and assume that

$$\lim_{x\to p} f(x) = a \text{ and } \lim_{x\to p} g(x) = b.$$

. Then,

1. $\lim_{x\to p}[f(x) \pm g(x)] = a \pm b.$
2. $\lim_{x\to p}[\lambda f(x)] = \lambda a.$
3. $\lim_{x\to p}[f(x)g(x)] = ab.$
4. $\lim_{x\to p} \dfrac{f(x)}{g(x)} = \dfrac{a}{b}, \text{ if } b \neq 0.$
5. $\lim_{x\to p} |f(x)| = |a|.$

3.7 **Continuous Functions and Properties** [1.5 Lectures]

(a) Definition of continuous function in terms of ϵ, δ, and in terms of open balls.

(b) **Theorem:** Every function is continuous at an isolated point.

(c) **Theorem** (Continuity of composite function): Let $(S, d_S), (T, d_T)$ and (U, d_U) be three metric spaces. Let $f : S \to T$ and $g : f(S) \to U$ be two functions. If f is continuous at a point $p \in S$ and g is continuous at $f(p)$, then the composite function $h = gof$ defined on S by the equation $h(x) = g(f(x))$ for all $x \in S$ is continuous at p.

(d) **Theorem** (Sequential criterion for continuity): Let (S, d_S) and (T, d_T) be two metric spaces, $f : S \to T$ be a function and assume that $p \in S$. Then f is continuous at p if, and only if, for every sequence $\{x_n\}$ of points of S converging to p, the sequence $\{f(x_n)\}$ of points of T converges to $f(p)$.

3.8 **Continuity and Inverse Images of Open and Closed Sets** [1.5 Lectures]

(a) Definition of inverse image (or pre-image) of a set and examples.

(b) **Theorem:** (Continuity and inverse image of open (or, closed) set):
Let $f : S \to T$ be a function from a metric space (S, d_S) to another metric space (T, d_T). Then f is continuous on S if, and only if, for every open (or, closed) set Y in T, the inverse image $f^{-1}(Y)$ is open (or, closed) in S.

(c) Examples to show the image of an open (or, a closed) set under a continuous map need not be open (or. closed).

3.9 **Continuous Functions on Compact Sets** [2 Lectures]

(a) **Theorem:** The image of a compact set under a continuous function is compact.

(b) Definition of a bounded vector valued function $\boldsymbol{f} : S \to \mathbb{R}^k$

(c) **Theorem:** Let $\boldsymbol{f} : S \to \mathbb{R}^k$ be a function from a metric space (S, d) to a Euclidean space $\mathbb{R}^k$. If $\boldsymbol{f}$ is continuous on a compact subset X of S, then $\boldsymbol{f}$ is bounded on X.

(d) Theorem: Let $\boldsymbol{f} : S \to \mathbb{R}^k$ be defined on a metric space (S, d_S) to the Euclidean space $\mathbb{R}^k$. If $\boldsymbol{f}$ is continuous on a compact subset $X$of S, then $\exists$ points p and q in X such that

$$\boldsymbol{f}(p) = \text{Infimum}\boldsymbol{f}(X) \text{ and}\boldsymbol{f}(q) = \text{Supremum}\boldsymbol{f}(X).$$

(e) **Corollary**: A continuous function attains its maximum and minimum on a compact set.

(f) **Theorem:** Let $f : S \to T$ be a function from a metric space (S, d_S) to another metric space (T, d_T). Assume that f is one-to-one on S, so that the inverse function f^{-1} exists. If S is compact and if f is continuous on S, then f^{-1} is continuous on $f(S)$.

(g) Example to show that compactness is essential in above theorem (d).

3.10 **Topological Mappings (Homeomorphisms)** [0.5 Lectures]

(a) Definition of topological mappings.

(b) Discussion about topological properties and isometry.

3.11 **Bolzano's Theorem** [1.5 Lectures]

(a) **Theorem** (Sign preserving property for continuous function):
Let f be defined on an interval S in $\mathbb{R}$. Assume that f is continuous at a point c in S and let $f(c) \neq 0$. Then, there exists an one ball $B(c, \delta)$ such that $f(x)$ has the same sign as $f(c)$ in $B(c, \delta) \cap S$.

(b) **Theorem** (Bolzano's theorem or the theorem for location of roots): Let f be a real-valued and continuous function on a compact interval $[a, b]$ in $\mathbb{R}$, and suppose that $f(a)$ and $f(b)$ have opposite signs, i.e. $f(a) \times f(b) < 0$. Then, there exists at least one point $c \in (a, b)$ such that $f(c) = 0$.

(c) **Theorem** (Intermediate value theorem): Let f be a real valued continuous function on a compact interval S in $\mathbb{R}$. Suppose that there are two points $a < b$ in S such that $f(a) \neq f(b)$. Then f takes every values between $f(a)$ and $f(b)$ in the interval (a, b).

3.12 **Uniform Continuity** [0.5 Lectures]

(a) Definition of uniform continuity and examples

(b) Non-uniform continuity condition

(c) **Theorem** Uniform continuity implies continuity, but the converse need not be true.

3.13 **Uniform Continuity and Compact Sets** [1 Lecture]

Theorem (Heine's theorem for uniform continuity on a compact set):
Let $f : S \to T$ be a function from one metric space (S, d_S) to another metric space (T, d_T). Let A be a compact subset of S and assume that f is continuous on A. Then, f is uniformly continuous on A.

3.14 **Exercises** [2 Lectures]

At least the following problems from Exercise on page 96, Chapter-4 of the book "Mathematical Analysis", Tom M Apostol II Edition (1987) should be discussed:

$4.2, 4.5, 4.6, 4.7, 4.9, 4.29, 4.30, 4.50, 4.51, 4.52.$

UNIT 4 FUNCTIONS OF BOUNDED VARIATION [13 Lecture Hrs]

4.1 **Review of Monotonic Functions** [1 Lecture]

(a) Definition of increasing (non-decreasing) and decreasing (non-increasing) functions
(b) Definition of strictly increasing and strictly decreasing functions
(c) Definition of monotonic functions
(d) Examples related to above definitions

4.2 **Properties of Monotonic Functions.** [1.5 Lectures]

(a) **Theorem:** If f is an increasing function defined on $[a,b]$ and let $x_0, x_1, \cdots, x_n$ be $n+1$ points such that $a = x_0 < x_1 < \cdots < x_n = b$, then

$$\sum_{k=1}^{n-1}[f(x_k^+) - f(x_k^-)] \le f(b) - f(a)$$

(b) Theorem: If f is monotonic on $[a,b]$, then the set of discontinuities of f is countable.

4.3 **Function of Bounded Variation (BV).** [3 Lectures]

(a) Definition of a partition of a compact interval.
(b) Definition of a function of BV on $[a,b]$.
(c) **Theorem:** If f is monotonic on $[a,b]$, then f is of BV on $[a,b]$.
(d) **Theorem:** If f is continuous on $[a,b]$ and if its derivative $f'(x)$ exists and is bounded in (a,b), then f is of BV on $[a,b]$.
(e) **Theorem:** If f is of BV on $[a,b]$, say $\sum |\Delta f_k| \le M$ for all partitions of $[a,b]$, then f is bounded on $[a,b]$. In fact, $|f(x)| \le |f(a)| + M$, $\forall x \in [a,b]$.
(f) Examples related to the functions of BV

i. The function $f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is of BV on [0, 1].

ii. The function $f(x) = \begin{cases} x \cos\left(\dfrac{\pi}{2x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous but not of BV on $[0,1]$.

iii. The function $f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous and is of BV on $[0,1]$.

iv. Boundedness of $f'(x)$ is not necessary for $f(x)$ to be of BV on $[0,1]$.

4.4 **Total Variation** [1.5 Lectures]

(a) Definition of total variation and its consequences.
(b) **Theorem:** Let f and g are each function of BV on $[a,b]$. Then, so are their sum, difference and product.
(c) The quotient f/g need not be a function of BV.
(d) **Theorem:** If f is of BV on $[a,b]$ and assume that f is of bounded away from zero. Then $g = 1/f$ is also of BV on $[a,b]$.

4.5 **Additive Property of Total Variation** [1.5 Lectures]

(a) **Theorem:** Let f be of BV on $[a,b]$ and assume that $c \in (a,b)$. Then f is of BV on $[a,c]$ and on $[c,b]$. And, we have $V_f(a,b) = V_f(a,c) + V_f(c,b)$.
(b) **Theorem:** If f is a function of BV on $[a,b]$, then it is so on every closed subinterval $[c,d]$ of $[a,b]$.

4.6 **Total Variation on [a, x] as a Function of x** [1 Lecture]

Theorem: Let f be a function of BV on $[a,b]$. Let V be defined on $[a,b]$ as follows:
$V(x) = V_f(a,x)$ for $a < x \le b$ and $V(a) = 0$. Then, both V and $V - f$ are increasing functions on $[a,b]$.

4.7 **Function of BV as a Difference of Two Increasing Functions** [1.5 Lectures]

(a) **Theorem:** f is of BV on $[a,b]$ if, and only if, f can be expressed as the difference of two increasing functions.
(b) **Theorem:** f is of BV on $[a,b]$ if, and only if, f can be expressed as the difference of two strictly increasing functions.
(c) Remarks: Representation of a function of BV as a difference of two increasing functions (or strictly increasing functions) is not unique.

4.8 **Continuous Function of Bounded Variations** [1 Lecture]

(a) **Theorem:** Let f be of BV on $[a,b]$. Define a function V on $[a,b]$ by $V(x) = V_f(a,x)$ for $x \in (a,b]$ and $V(a) = 0$. Then, every point of continuity of f is also a point of continuity of V, and conversely.

(b) **Theorem:** Let f be a continuous function on $[a,b]$. Then f is of BV on $[a,b]$ if, and only if, f can be expressed as the difference of two increasing continuous functions.

4.9 **Exercises.** [1 Lecture]

At least the following problems from Exercise on page 137, Chapter-6 of the book "Mathematical Analysis", Tom Apostol II Edition (1987) should be discussed:
6.1, 6.3.
Note: If time permits, one can give a list of simple problems for exercise to the students from other books like 'Introduction to Mathematical Analysis' by Amritava Gupta, 'The Elements of Real Analysis' by Robert G. Bartle etc.

Unit 5 RIEMANN STIELTJES (R-S) INTEGRATION [18 Lecture Hrs]

5.1 **Review of Riemann Integrals** [0.5 Lectures]

(a) Definition of partition, Norm of a partition, Refinement of a partition, Lower, Upper and Riemann Riemann sums of a bounded function.

(b) Definition of upper and lower integrals, Riemann integral.

(c) **Theorem:** (Statement only) Necessary and sufficient conditions for the existence of Riemann integrability

5.2 **Riemann-Stieltjes Integrals** [0.5 Lectures]

(a) Definition of R-S sum of a bounded function f w.r.to α on $[a,b]$.

(b) Definition of R-S integral of a bounded function f w.r.to α on $[a,b]$.

5.3 **Linear Properties of R-S Integrals** [1 Lecture]

(a) **Theorem** (Linearity on integrand): If $f \in \mathcal{R}(\alpha), g \in \mathcal{R}(\alpha)$ on $[a,b]$, then
$c_1 f + c_2 g \in \mathcal{R}(\alpha)$ on $[a,b]$, where c_1 and c_2 are two constants, and also we have

$$\int_a^b (c_1 f + c_2 g) d\alpha = c_1 \int_a^b f d\alpha + c_2 \int_a^b g d\alpha$$

(b) **Theorem** (Linearity on integrator): If $f \in R(\alpha), f \in \mathcal{R}(\alpha)$ on $[a,b]$, then
$f \in \mathcal{R}(c_1\alpha + c_2\beta)$ on $[a,b]$, where c_1 and c_2 are two constants and also we have

$$\int_a^b f d(c_1\alpha + c_2\beta) = c_1 \int_a^b f d\alpha + c_2 \int_a^b f d\beta$$

(c) **Theorem** (Additive property of R-S integral): Assume that $c \in (a,b)$. If any two of the three integrals in (1) exist, then the third also exists and we have

$$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha \tag{1}$$

5.4 **Integration by Parts** [1 Lecture]

(a) **Theorem** (Integration by parts):
If $f \in \mathcal{R}(\alpha)$ on $[a,b]$, then $\alpha \in \mathcal{R}(f)$ on $[a,b]$. Moreover, we have

$$\int_a^b f(x) d\alpha(x) + \int_a^b \alpha(x) df(x) = f(b)\alpha(b) - f(a)\alpha(a).$$

(b) Examples to support the above formula.

5.5 **Change of Variables in R-S Integrals** [1 Lecture]

Theorem (R-S integrability of composite functions):

(a) Let $f \in R(\alpha)$ on $[a,b]$ and let g be a strictly monotonic continuous function defined on interval $[c,d]$.

(b) Let $a = g(c)$ and $b = g(d)$.

(c) Let h and β be the composite functions defined by

$$h(x) = f[g(x)] \text{ and } \beta(x) = \alpha[g(x)]\, \forall x \in S.$$

Then $h \in \mathcal{R}(\beta)$ on S and we have $\int_a^b f d\alpha = \int_c^d h d\beta$. i.e.,

$$\int_{g(c)}^{g(d)} f(x) d\alpha(x) = \int_c^d f[g(x)] d\{\alpha[g(x)]\}.$$

5.6 **Reduction to a Riemann Integral.** [1 Lecture]

(a) **Theorem** (Relation between Riemann and Reimann-Stieltjes integrals):
Let $f \in R(\alpha)$ on $[a, b]$ and assume that α has a continuous derivative α' on $[a, b]$. Then, the Riemann integral $\int_a^b f(x)\alpha'(x)dx$ exists and we have

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx.$$

In other words, R-S integral of f w.r.t. α on $[a, b]$ is equal to the Riemann integral of $f\alpha'$ on $[a, b]$.

(b) Example to support the above theorem.

5.7 **Step Function as Integrators** [0.5 Lectures]

(a) Definition of Step Function

(b) **Theorem:** (Step functions as integrator) (Statement only)
Let $c \in (a, b)$ and define α on $[a, b]$ as follows. The values of $\alpha(a), \alpha(b), \alpha(c)$ are arbitrary and $\alpha(x) = \alpha(a)$ if $a \le x < c$ and $\alpha(x) = \alpha(b)$ if $a < x \le c$. Let f be defined on $[a, b]$ in such a way that at least one of f or α is continuous at c from the left and at least one is continuous at c from the right at c. Then, $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and we have $\int_a^b f d\alpha = f(c)[\alpha(c^+) - \alpha(c^-)]$.

(c) Examples:

i. The existence of the value of R-S integral can also be affected by changing the value of the integrand f at a single point.

ii. In a Riemann integral $\int_a^b f(x)dx$, the values of f can be changed at finite number of points without affecting either the existence or the value of the integral.

5.8 **Reduction of R-S Integral to a Finite sum.** [1 Lecture]

(a) **Theorem** (Reduction of a R-S integral to a finite sum) (Statement only)

(b) **Theorem** (Reduction of finite Sum to R-S integral) (Statement only)

5.9 **Monotonically Increasing Integrators, Upper & Lower Integrals** [1.5 Lectures]

(a) Definition of upper and lower Stieltjes sums and their integrals

(b) **Theorem** (Refinement of partition): Assume that $\alpha \uparrow$ on [a, b]. Then,

i. $P, P' \in \mathcal{P}[a, b]$ and $P' \supseteq P \implies U(P', f, \alpha) \le U(P, f, \alpha)$ and $L(P', f, \alpha) \ge L(P, f, \alpha)$.

ii. For any two partitions P_1 and P_2 of $[a, b], L(P_1, f, \alpha) \ge U(P_2, f, \alpha)$.

(c) **Theorem** (Comparison of lower and upper integrals): Assume that $\alpha \uparrow$ on $[a, b]$. Then $\underline{I}(f, \alpha) \le \overline{I}(f, \alpha)$.

(d) Example to show that $\underline{I}(f, \alpha) \le \overline{I}(f, \alpha)$.

5.10 **Riemann's Conditions for R-S Integral** [1 Lecture]

(a) Definition of Riemann's condition.

(b) **Theorem** (Riemann's condition): Statement and proof

5.11 **Comparison Theorems** [1.5 Lectures]

(a) **Theorem** (Order preserving property in R-S integral): Assume that $\alpha \uparrow$ on $[a, b]$, $f, g \in \mathcal{R}(\alpha)$ on $[a, b]$ and if $f(x) \le g(x) \forall x \in [a, b]$, then we have
$\int_a^b f(x)d\alpha(x) \le \int_a^b g(x)d\alpha(x)$.

(b) **Theorem:** Assume that $\alpha \uparrow$ on $[a, b]$ and $f \in \mathcal{R}(\alpha)$ on $[a, b]$, then $|f| \in \mathcal{R}(\alpha)$ on $[a, b]$, and we have
$\left|\int_a^b f(x)d\alpha(x)\right| \le \int_b^a |f(x)|d\alpha(x)$.

(c) **Theorem** (R-S integrability of the square function): Assume that $\alpha \uparrow$ on [a, b] and $f \in \mathcal{R}(\alpha)$ on $[a, b]$, then $f^2 \in \mathcal{R}(\alpha)$ on $[a, b]$.

(d) **Theorem** (R-S Integrability of the product): Assume that $\alpha \uparrow$ on [a, b]. If
$f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$, then the product $fg \in \mathcal{R}(\alpha)$ on $[a, b]$. In other words, the product of two R-S integrable functions is again R-S integrable.

5.12 **Necessary and Sufficient Conditions for R-S Integrability** [1.5 Lectures]

(a) **Theorem:** (A Sufficient condition for R-S integrals): If f is continuous on $[a, b]$ and if α is of BV on $[a, b]$, then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

(b) **Theorem** (A sufficient condition for R-S integrals): Let f be of BV on $[a, b]$ and α is continuous on $[a, b]$, then $f \in R(\alpha)$ on $[a, b]$.

(c) **Theorem:** (A sufficient condition for Riemann integrals): Each of the following conditions is sufficient for the existence of the Riemann integral $\int_b^a f(x)dx$:

   i. f is continuous on $[a, b]$

   ii. f is of BV on $[a, b]$

(d) **Theorem** (Necessary condition for existence of R-S integrals): Assume that $\alpha \uparrow$ on $[a, b]$ and $c \in (a, b)$. Assume that both f and α are discontinuous at $x = c$ from the right, i.e. assume that there exists an $\epsilon > 0$ such that $\forall \delta > 0$, there are values of x and y in the interval $(c, c+\delta)$ satisfying

$$|f(x) - f(y)| \geq \epsilon \text{ and } \alpha(y) - \alpha(c) \geq \epsilon.$$

Then, the integral $\int_a^b f(x)d\alpha(x)$ cannot exist. Similarly, if both f and α are discontinuous at $x = c$ from the left, then $\int_a^b f(x)dx$ cannot exist.

5.13 **Mean Value Theorems (MVTs) for R-S Integrals.** [1.5 Lectures]

(a) **Theorem:** (First MVT for R-S integral):
Assume that $\alpha \uparrow$ on $[a, b]$ and $f \in R(\alpha)$ on $[a, b]$. Let M and m are the supremum and infimum of the set $\{f(x) : x \in [a, b]\}$. Then $\exists$ a real number c satisfying

$$m \leq c \leq M \text{ such that } \int_a^b f(x)d\alpha(x) = c\int_a^b d\alpha(x) = c[\alpha(b) - \alpha(a)]$$

In particular, if f is continuous on $[a, b]$, then $c = f(x_0)$ for some x_0 in $[a, b]$.

(b) The restriction that c is a point of the closed interval is essential in the statement of the First MVT.

(c) **Theorem:** (Second mean value theorem for R-S integral):
Assume that α is continuous and $f \uparrow$ on $[a, b]$. Then $\exists$ a point $x_0 \in [a, b]$ such that

$$\int_a^b f(x)d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)$$

(d) Statement of the corresponding theorem for Riemann integral.

5.14 **Integral as the Function of the Interval.** [1 Lecture]

(a) Definition of integral function.

(b) **Theorem:** Let α be of BV on $[a, b]$ and assume that $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Define F on $[a, b]$ by an equation

$$F(x) = \int_a^x f(x)d\alpha(x), \text{ where } x \in [a, b].$$

Then, we have

   i. F is of BV on $[a, b]$.

   ii. Every point of continuity of α is also a point of continuity of F.

   iii. If $\alpha \uparrow$ on $[a, b]$, then the derivative of $F(x)$ exists at point $x \in (a, b)$, where $\alpha'(x)$ exists and f is continuous. For such x, we have $F'(x) = f(x)\alpha'(x)$.

(c) Theorem of Integrators of BV (Statement only).

(d) **Theorem** (Conversion of a Riemann integral of the product $f \cdot g$ into an R-S integral): Let $f \in \mathcal{R}$ on $[a, b]$, $g \in \mathcal{R}$ on $[a, b]$ and let

$$F(x) = \int_a^x f(t)dt, \quad G(x) = \int_a^x g(t)dt \text{ if } x \in [a, b].$$

Then F and G are continuous functions of BV on $[a, b]$. Also $f \in \mathcal{R}(G)$ and $g \in \mathcal{R}(F)$ on $[a, b]$, and we have

$$\int_a^b f(x)g(x)dx = \int_a^b f(x)dG(x) = \int_a^b g(x)dF(x)$$

5.15 **Fundamental Theorems of Integral Calculus** [2 Lectures]

(a) Definition of primitive (or antiderivative) of a function with examples

(b) **Theorem** (First fundamental theorem of integral calculus): Let $f \in \mathcal{R}$ and be continuous on $[a, b]$. Let $F(x) = \int_a^x f(t)dt$ for all $x \in [a, b]$, then the derivative $F'(x)$ exists and $F'(x) = f(x)$. In other words, the integral function $F(x)$ of a continuous function f is differentiable for all $x \in [a, b]$ and satisfies $\dfrac{d}{dx}[F(x)] = f(x)$.

(c) **Theorem** (Second fundamental theorem of integral calculus):
Assume that $f \in \mathcal{R}$ on $[a, b]$ and let F be a function defined on $[a, b]$ such that the derivative $F'(x)$ exists at each point $x \in (a, b)$ and $F'(x) = f(x) \, \forall x \in (a, b)$. At the end points, assume that $F(a^+)$ and $F(b^-)$ exist and satisfy

$$F(a) - F(a^+) = F(b) - F(b^-).$$

Then, we can have $\int_b^a f(x)dx = \int_a^b F'(x)dx = F(b) - F(a)$.
In other words, if $F : [a, b] \to \mathbb{R}$ is a primitive of $f : [a, b] \to \mathbb{R}$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

5.16 **Exercise** [1.5 Lectures]
At least the following problems from Exercise on page 174, Chapter-7 of the book "Mathematical Analysis", Tom Apostol II Edition (1987) should be discussed:
$7.1, 7.2, 7.3, 7.11, 7.12, 7.13$.

UNIT 6 MULTIVARIABLE DIFFERENTIATION [13 Lecture Hrs]

6.0 **Review** [0.5 Lectures]

(a) Review of derivative of vector valued function $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^n$ and its limitations.
(b) Review of partial derivatives of function $f : \mathbb{R}^n \to \mathbb{R}$ and its limitations.

6.1 **Linear Operators and Matrix Representations** [0.5 Lectures]

(a) Review of the process of finding an $m \times n$ matrix $\mathcal{M}(\boldsymbol{T})$ of scalars for the linear operator $\boldsymbol{T} : \mathbb{R}^n \to \mathbb{R}^m$.
(b) Matrix of the composite map: $\mathcal{M}(\boldsymbol{S}o\boldsymbol{T}) = \mathcal{M}(\boldsymbol{S})\mathcal{M}(\boldsymbol{T})$(Statement and its meaning).

6.2 **Directional Derivatives** [1 Lecture]

(a) Definition of directional derivative for functions $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ in the direction given by $\boldsymbol{u}$.
(b) Consequence of definition (a).

6.3 **Directional Derivatives and Continuity** [1 Lecture]

(a) Example to show the existence of partial derivatives does not imply the existence of directional derivatives $\boldsymbol{f}'(\boldsymbol{c}; \boldsymbol{u})$ in all the directions given by $\boldsymbol{u}$.
(b) Example to show the existence of $\boldsymbol{f}'(\boldsymbol{c}; \boldsymbol{u})$ in all $\boldsymbol{u}$ does not imply the continuity of $\boldsymbol{f}$ at $\boldsymbol{c}$.

6.4 **Total Derivative** [2 Lectures]

(a) Definition of total derivative with review of error function in one dimensional case.
(b) **Theorem** (Equality of total derivative and directional derivative): If $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\boldsymbol{c}$ in $S \subseteq \mathbb{R}^n$ with total derivative $\boldsymbol{T_c}$, then the directional derivative $\boldsymbol{f}'(\boldsymbol{c}; \boldsymbol{u})$ exists for every $\boldsymbol{u}$ in $\mathbb{R}^n$ and we have $\boldsymbol{f}'(\boldsymbol{c}; \boldsymbol{u}) = \boldsymbol{T_c}(\boldsymbol{u})$.
(c) **Theorem** (Uniqueness of total derivative): Total derivative of a differentiable function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ if exists, is unique.
(d) **Theorem** (Differentiability implies continuity): Let $S \subseteq \mathbb{R}^n$ and if $\boldsymbol{f} : S \to \mathbb{R}^m$ is differentiable at $\boldsymbol{c}$. Then, $\boldsymbol{f}$ is continuous at $\boldsymbol{c}$.

6.5 **Total Derivatives Expressed in terms of Partial Derivatives** [1 Lecture]

(a) **Theorem:** A function which is differentiable at point $\boldsymbol{c}$ admits first order partial derivative at that point.
(b) Proof of the relation:

$$\boldsymbol{f}'(\boldsymbol{c})(\boldsymbol{v}) = \sum_{i=1}^{n} v_k D_k \boldsymbol{f}(\boldsymbol{c})$$

6.6 **The Jacobian Matrix** [1 Lecture]

(a) Derivation of Jacobian matrix (Matrix in connection with total derivative)
Statement: Let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ be a function differentiable at each point $\boldsymbol{c} \in \mathbb{R}^n$ with total derivative $\boldsymbol{T} = \boldsymbol{f}'(\boldsymbol{c})$. Then $\boldsymbol{T}$ can be represented by an $m \times n$ matrix with entries of the components $\boldsymbol{Df}(\boldsymbol{c})$.
(b) Derivation of the inequality

$$\|\boldsymbol{f}'(\boldsymbol{c})(\boldsymbol{v})\| \le M\|\boldsymbol{v}\|, \text{ where } M = \sum \|\nabla f_k(\boldsymbol{c})\|$$

6.7 **The Mean Value Theorem (MVT) for Differentiable Functions** [1 Lecture]

(a) Review of one dimensional MVT and definition of line segment joining $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^n$.

(b) MVT on Multivariable Calculus for Differentiable Functions $\boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^m$

(c) Application of MVT: Statements only (In the proof, the notion of connectedness is used, which is not introduced in our course).

6.8 **Sufficient Condition for Differentiability** [1 Lecture]

Theorem (Sufficient condition for differentiability) (Statement only): Let S be the subset of $\mathbb{R}^n$ and assume $\boldsymbol{f}: S \to \mathbb{R}^n$ be a function. Assume that one of the partial derivatives $D_1\boldsymbol{f}, D_2\boldsymbol{f}, \cdots D_n\boldsymbol{f}$ exists at $\boldsymbol{c} \in S$ and that the remaining $n-1$ partial derivatives exist in some $n-$ball $B(\boldsymbol{c})$ and are continuous at $\boldsymbol{c}$. Then $\boldsymbol{f}$ is differentiable at $\boldsymbol{c}$.
Some examples related to this theorem for up to $n = 3$.

6.9 **Sufficient Conditions for Equality of Mixed Partial Derivatives** [2 Lectures]

(a) Notation $D_{r,k}\boldsymbol{f} = D_r(D_k(\boldsymbol{f})) = \dfrac{\partial^2 \boldsymbol{f}}{\partial x_r \partial x_k}$.

(b) Example to show that mixed partial derivatives $D_{1,2}f(x,y)$ and $D_{2,1}f(x,y)$ are not necessarily equal, i.e. $f_{xy} \neq f_{yx}$.

(c) **Theorem:** (A sufficient condition for equality mixed partial derivatives)
Young's Theorem: Assume that $\boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^m$. If both partial derivatives $D_r\boldsymbol{f}$ and $D_k\boldsymbol{f}$ exist in an n-ball $B(\boldsymbol{c};\delta)$ and if both are differentiable at $\boldsymbol{c}$, then $D_{r,k}\boldsymbol{f}(\boldsymbol{c}) = D_{k,r}\boldsymbol{f}(\boldsymbol{c})$.

(d) **Theorem** (Schwarz's Theorem) (Statement only): If both partial derivatives $D_r\boldsymbol{f}$ and $D_k\boldsymbol{f}$ exist in an open $n-$ball $B(\boldsymbol{c})$ and if both $D_{r,k}\boldsymbol{f}$ and $D_{k,r}\boldsymbol{f}$ are continuous at $\boldsymbol{c}$. Then $D_{r,k}\boldsymbol{f}(\boldsymbol{c}) = D_{k,r}\boldsymbol{f}(\boldsymbol{c})$.

6.10 **Exercises** [2 Lectures]

At least the following problems from Exercise on page 362, Chapter-12 of the book "Mathematical Analysis", Tom M. Apostol II Edition (1987) should be discussed:
$12.3, 12.4, 12.7, 12.8, 12.12, 12.24$.

UNIT 7 SEQUENCES AND SERIES OF FUNCTIONS [13 Lecture Hrs]

7.1 **Sequence of Functions** [5 Lectures]

(a) Introduction of sequences of functions

(b) Definition of limit function and pointwise convergence

(c) Examples:
  i. Sequence of continuous functions with a discontinuous limit function
  ii. Sequence of functions for which $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$
  iii. Sequence of differentiable functions $\{f_n\}$ with limit 0 for which $\{f_n'\}$diverges

(d) Definition of uniform convergence and uniform boundedness.

(e) Geometrical interpretation of uniform convergence

(f) Proof of the fact that uniform convergence of a sequence of functions implies its pointwise convergence but converse may not be true.

(g) Cauchy's criterion for uniform convergence for sequence and its proof.

(h) Theorems related to uniform convergence and continuity:
  i. **Theorem** (Uniform convergence preserves continuity): If $f_n \to f$ uniformly on a set S and if each f_n is continuous on S, then f is continuous on S.
  ii. **Theorem** (Dini's uniform convergence theorem):
  If $\{f_n\}$ is a sequence of continuous functions converging pointwise to a continuous function f on a compact set $S \subseteq \mathbb{R}^n$, and if $\{f_n(x)\}$ is monotone decreasing, then $f_n \to f$ (uniformly) on S

(i) Uniform convergence and differentiation for sequence
Theorem (Statement only): Assume that $\{f_n\}$ is a real valued function having a finite derivative at each point of (a,b). Assume that for at least one point x_0 in (a,b), the sequence $\{f_n(x_0)\}$ converges. Assume further that there exists a function g such that $f_n' \to g$ uniformly on (a,b). Then
  i. there exists a function f such that $f_n \to f$ uniformly on (a,b).
  ii. for each x in (a,b), the derivative $f'(x)$ exists and $f'(x) = g(x)$.

(j) Uniform convergence and integration for sequence:
Theorem (Statement only): Let $\{f_n\}$ be a sequence of continuous functions on $[a,b]$. If $f_n \to f$ uniformly on $[a,b]$, then
$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

7.2 **Series of Functions** [6 Lectures]

(a) Definition of uniform convergence of series of functions.

(b) The theorem of Cauchy's condition for uniform convergence of series and its poof.

(c) Statements and proof of Weierstrass M-test, Dirichlet's test, Abel's test, and their applications to test the uniform convergence of series of functions.

(d) The theorem on continuity of the limit function of a uniformly convergent series.
Theorem: Assume that $\sum f_n(x) = f(x)$ (uniformly on S). If each f_n is continuous at a point c of S, then f is also continuous at c.

(e) Uniform convergence and differentiation of Series
Theorem (Statement only):
Assume that each f_n is a real valued function defined on (a, b) such that the derivative $f_n'(x)$ exists for each $x \in (a, b)$. Assume that for at least one point x_0 in (a, b), the series $\sum f_n(x_0)$ converges. Assume further that there exists a function g such that $\sum f_n'(x) = g(x)$ (uniformly) on (a, b). Then
(a) there exists a function f such that $\sum f_n(x) = f(x)$ (uniformly) on (a, b).
(b) if $x \in (a, b)$, the derivative $f'(x)$ exists and equals to $\sum f_n'(x)$.

(f) Uniform convergence and integration for series:
Theorem (Term by term Integration of Series):
Let $\sum_{n=1}^{\infty} f_n$ be a series of continuous functions on $[a, b]$. If $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on $[a, b]$, then

$$\sum_{n=1}^{\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx \quad \forall x \in [a, b].$$

7.3 Exercise [2 Lectures]

At least the following problems from Exercise on page 247, Chapter-9 of the book "Mathematical Analysis", Tom M. Apostol II Edition (1987) should be discussed:
$9.1, 9.3, 9.4, 9.5, 9.6, 9.11, 9.12, 9.13, 9.18, 9.22$

UNIT 8 IMPROPER INTEGRALS [18 Lecture Hrs]

8.1 **Classification of Improper Integrals with Examples** [1 Lecture]

8.2 **Improper Integrals of First Kind** [9 Lectures]

(a) Definition of convergence and divergence

(b) Application of the fundamental theorem of integral calculus.

(c) The convergence or the divergence of the improper integrals of the functions:

$$(a) \quad \frac{1}{x^2} \text{ on } (1, \infty) \qquad\qquad (b) \quad \sin x \text{ on } (0, \infty)$$

$$(c) \quad \frac{1}{x^\alpha} \text{ on } (a, \infty), \text{ where } a > 0 \quad (d) \quad e^{-rx} \text{ on } (0.\infty), \text{ where } r > 0$$

(d) Geometrical interpretation of convergent improper integral of the first kind.

(e) Analogy of the improper integral $\int_a^\infty f(x)dx$ with infinite series $\sum_{n=1}^\infty a_n$

(f) The theorem on the Cauchy's criterion for convergence
Theorem (Cauchy's Criterion): Let f be integrable over $[a, t]$ for all $t \geq a$. Then the integral $\int_a^\infty f(x)dx$ converges if, and only if, for every $\epsilon > 0$, there exists a point $t_0 > a$ such that for any two points t'', t' satisfying $t'' > t' \geq t_0$ implies

$$\left| \int_{t'}^{t''} f(x)dx \right| < \epsilon.$$

(g) Linear properties of improper integral:
Theorem: If $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ exist, then $\int_a^\infty (f(x) \pm g(x))dx$ exists and we have

$$\int_a^\infty (f(x) \pm g(x))dx = \int_a^\infty f(x)dx \pm \int_a^\infty g(x)dx.$$

Theorem: If $\int_a^\infty f(x)dx$ exists, then for every $c \in \mathbb{R}$, $\int_a^\infty f(x)dx$ exists and

$$\int_a^\infty cf(x)dx = c\int_a^\infty f(x)dx.$$

(h) A necessary and sufficient condition for the convergence of the improper integral: **Theorem:** Let $f(x) \geq 0$ for all $x \geq a$ and let f be integrable over $[a, t]$ for all $t \geq a$. Then the integral $\int_a^\infty f(x)dx$ converges if, and only if, the set
$\{I_1 = \int_a^t f(x)dx,\ t \geq a\}$ is bounded. In this case,

$$\int_a^\infty f(x)dx = \sup\left\{ I_t : I_t = \int_a^t f(x)dx \right\}.$$

(i) The comparison test, and the limit comparison test to determine the convergence or divergence of some improper integrals.
Theorem (Comparison test for convergence and divergence): Let f and g be two integrable functions over $[a, t]$ for all $t \geq a$ and if $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then,
   i. if the integral $\int_a^\infty g(x)dx$ converges, then the integral $\int_a^\infty f(x)dx$ also converges.
   ii. if the integral $\int_a^\infty f(x)dx$ diverges, then the integral $\int_a^\infty g(x)dx$ also diverges.

Theorem (Limit comparison test): Let $g(x) > 0$ and if $lim_{x\to\infty}\dfrac{f(x)}{g(x)} = L$, finite and nonzero, then both integrals converge or diverge together. Also if $\lim_{x\to\infty}\dfrac{f(x)}{g(x)} = 0$, and $\int_a^\infty g(x)dx$ converges, then the integral $\int_a^\infty f(x)dx$ also converges, and if $lim_{x\to\infty}\frac{f(x)}{g(x)} = \infty$ and $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ also diverges.
Theorem (Limit comparison test for convergence): Let $0 \leq f$ be integrable over $[a, t]$ for all $t \geq a$ and for $p > 1$, $x^p f(x) = L$ (finitely exists), then the integral
$\int_a^\infty f(x)dx$ converges.
Theorem: Let $f(x)$ be integrable over $[a, t]$ for all $t \geq a$ and $\lim_{x\to\infty} xf(x) = L \neq 0$ (or$= \pm\infty$). Then, the integral $\int_a^\infty f(x)dx$ diverges.
Theorem (Statement only): Let f be integrable over $[a, t]$, for all $t \geq a$ and $\lim_{x\to\infty} xf(x) = L$ or $\pm\infty$. If $L \neq 0$, then $\int_a^\infty f(x)dx$ diverges and if $L = 0$, the test fails.

(j) Definitions of absolute and conditional convergence of $\int_a^\infty f(x)dx$

(k) Theorem asserting that absolute convergence implies its convergence:
Theorem: If the integral $\int_a^\infty |f(x)|dx$ converges, then $\int_a^\infty f(x)dx$ also converges.
i.e., every absolutely convergent improper integral is necessarily convergent.

(l) Theorem on absolute convergence of the improper integral of a product:
Theorem (Absolutely integrability of the product): If $\int_a^\infty f(x)dx$ converges absolutely and g is bounded on$[a,\infty)$, then the product $f(x)g(x)$ is also absolutely integrable over $[a,\infty)$.

(m) Dirichlet's test and Abel's test for the convergence of the improper integral of a product:
Theorem (Dirichlet's test): Assume that (i) f is integrable over $[a,t]$ for all $t \geq a$ and there exists a constant $M > 0$ such that

$$\forall t \geq a, \quad \left|\int_a^t f(t)dt\right| \leq M.$$

(ii) $g(x)$ is monotonic decreasing to 0 as $x \to \infty$, *i.e.*, $g(x) \to 0$ as $x \to \infty$. Then the integral $\int_a^\infty f(x)g(x)dx$ is convergent.

Theorem (Abel's test): Assume that (i) the integral $\int_a^\infty f(x)dx$ converges
(ii) $g(x)$ is monotonic and bounded on $[a,\infty)$ *i.e.*, $\exists$ a constant $M > 0$ such that $|g(x)| \leq M$. Then, the integral $\int_a^\infty f(x)g(x)dx$ is convergent.

(n) Variants of the improper integral of the first kind.

(o) Definition of the Cauchy principle value of the integral over $\mathbb{R}$. To show the fact that the existence of the Cauchy principal value may not always imply the convergence of the integral $\int_{-\infty}^\infty f(x)dx$.

8.3 **Improper Integral of the Second Kind** [5 Lectures]

(a) Definition of convergence and divergence.

(b) Reduction of the improper integral of the second kind into that of the first kind.

(c) Geometrical interpretation of the convergent improper integral of the second kind $\int_a^b f(x)dx$.

(d) Theorem on Cauchy's criterion for convergence:
Theorem (Cauchy's criterion) (Statement only): Assume that f be integrable over $[t,b]$ for all $t \in (a,b]$. Then, the integral$\int_a^b f(x)dx$ converges if, and only if, for every $\epsilon > 0$, there exists $t_0 \in (a,b]$ such that $a < t' < t" \leq t_0$ implies

$$\left|\int_{t'}^b f(x)dx - \int_{t''}^b f(x)dx\right| = \left|\int_{t'}^{t''} f(x)dx\right| < \epsilon$$

(e) The comparison test, and the limit comparison test to determine convergence or divergence of some improper integrals:
Theorem (Comparison test for integrands) (Statement only): Let f, g be two integrable functions over $[t,b]$ for all $t \in (a,b]$ and if $0 \leq f(x) \leq g(x)$ for all $x \in (a,b]$.
(i) If the integral $\int_a^b g(x)dx$ converges, then the integral $\int_a^b f(x)dx$ also converges.
(ii) If the integral $\int_a^b f(x)$ diverges, then the integral $\int_a^b g(x)dx$ also diverges.
Remarks: Above comparison test can be formulated in limit form as follows:
Theorem (Statement only) (Comparison test for integrands in limit): Let f, g be two functions integrable over $(t,b]$ for all $t \in (a,b]$ and if

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L,$$

where L is finite. Then, the two integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge and diverge together.
Theorem (Statement only) (Limit comparison test for convergence and divergence): Let $0 \leq f(x)$ be bounded and integrable over $[t,b]$ for all $t \in (a,b]$. Then (i) If $0 < p < 1$ and $\lim_{x\to a^+}(x-a)^p f(x)$ exists and say L, then $\int_a^b f(x)dx$ converges. (ii) If $p \geq 1$ and $\lim_{x\to a^+}(x-a)^p f(x)$ exists and non zero $(\pm\infty)$ then the integral diverges.

(f) The variants of the improper integral of the second kind.

(g) Definition of the Cauchy principle value of the improper integral of the second kind. Example to show that the existence of the Cauchy principal value may not imply the convergence of the integral.

8.4 **Exercise** [3 Lectures]
From the book "Advanced Calculus by D.V. Widder", Chapter 10, related problems should be discussed.

Unit 9 COMPLEX NUMBERS AND FUNCTIONS [13 Lecture Hrs]

9.1 **Review, Different forms and Roots of Complex Numbers**

(a) Review of definition of complex number, Sum and product of complex numbers, Basic algebraic properties of complex numbers, Modulus and conjugate of complex numbers, Triangle inequality, Polar form of complex numbers, Related Examples.
[2.5 Lectures]

(b) Products, quotients and powers in exponential form, Arguments of products and quotients
Examples $1, 2$ on Page No. 20 and Examples $1, 2$ on Page No. $21, 22$ [2 Lectures]

(c) Roots of complex numbers.
Examples 1, 2, 3 on Page No. 27, 28, 29 [2 Lectures]

(d) Region in the complex plane:
Definition of ϵ-neighbourhood, Deleted neighbourhood, Interior, exterior and boundary points, Open and closed sets, Domain and region, Bounded and unbounded sets. [1.5 Lectures]

9.2 **Function of a Complex Variable and Mappings** [3 Lectures]

(a) Definition of function of a complex variable
(b) Definition of mapping with examples
(c) Mapping by $w = z^2$. Example 1, on Page No. 39
(d) Mapping by $w = e^z$, Example $1, 2$ on Page No. 42 and 43.

9.3 **Exercises** [2 Lectures]
At least the following problems from the following pages of the book Complex Variables and Applications, 8th edition, by James Ward Brown & Ruel V. Churchill should be discussed:
Page No. 5 : 1, 2, 3, 4; Page No. 8 : 1, 2, 3 ; Page No. 12: 1, 3, 4, 5, 6 ; Page No. 14: 1, 2, 7, 13, 15 ; Page No. 22-23: 1, 2, 3, 5, 6; Page No. 29-30: 1, 2, 3, 8 Page No. 37: 1, 2, 3.

UNIT 10 ANALYTIC FUNCTIONS [13 Lecture Hrs]

10.1 **Limits and Continuity** [3 Lectures]

(a) Definition of limit of a function $f(z)$ of a complex variable z as $z \to z_0$.
Examples 1 and 2, Page No. 46, 47.

(b) Theorems on limits.
i. **Theorem:** Suppose that $f(z) = u(x, y)+iv(x, y)$, $(z = x+iy)$ and $z_0 = x_0+iy_0, w_0 = u_0+iv_0$ Then,
$$\lim_{z\to z_0} f(z) = w_0 \text{ if, and only if, } \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0, \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$
ii. **Theorem** (Statement only): Suppose that
$$\lim_{z\to z_0} f(z) = w_0, \quad \lim_{z\to z_0} F(z) = W_0. \quad \text{Then,}$$
$$\lim_{z\to z_0} (f(z) + F(z)) = w_0 + W_0, \qquad \lim_{z\to z_0} (f(z)F(z)) = w_0W_0$$
and if $W_0 \neq 0$
$$\lim_{z\to z_0} \frac{f(z)}{F(Z)} = \frac{w_0}{W_0}$$

(c) Limit involving the point at infinity:
Theorem (Statement only): If z_0 and w_0 are points in the z and w planes respectively, then
$$\lim_{z\to z_0} f(z) = \infty \text{ if, and only if, } \lim_{z\to z_0} \frac{1}{f(z)} = 0$$
and
$$\lim_{z\to \infty} f(z) = w_0 \text{ if, and only if, } \lim_{z\to 0} \frac{1}{f(z)} = w_0.$$
Moreover,
$$\lim_{z\to z_0} f(z) = \infty \text{ if, and only if, } \lim_{z\to 0} \frac{1}{f(\frac{1}{z})} = 0.$$

Examples on Page No. 52.

(d) Continuity:
Definition of continuity of a function $f(z)$ of a complex variable at $z = z_0$
Theorem: The composition of continuous functions is itself continuous.
Theorem: If a function $f(z)$ is continuous and non-zero at a point z_0, then $f(z) \neq 0$ throughout some neighbourhood of that point.

10.2 **Complex Differentiation** [4 Lectures]

(a) Definition of derivative of a function of complex variable.
Examples $1, 2, 3$ on Page No. $57, 58$.

(b) Differentiation formulae (Statement only, as the proofs are similar to those in real variables).

(c) **Theorem:** Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first order partial derivatives of u and v must exists at (x_0, y_0) and satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$
Also, $f'(z_0) = u_x + iv_x$, where these partial derivatives are to be evaluated at (x_0, y_0).
Examples 1 and 2, on Page No. 65, 66.

(d) Sufficient condition for differentiability:
Theorem: Let $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ϵ-nbhd of a point $z_0 = x_0+iy_0$, and suppose that (i) the first order partial derivatives of the functions u and v with respect to x and y exists everywhere in the neighborhood; (ii) those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ at (x_0, y_0). Then $f'(z_0)$ exists and its value being $f'(z_0) = u_x + iv_x$ where the right hand side is to be evaluated at (x_0, y_0).
Examples 1,2 of Page No. 68

(e) Cauchy-Riemann equations in polar form:
Theorem: Let the function $f(z) = u(r, \theta) + iv(r, \theta)$ be defined throughout some ϵ-neighbourhood of a non zero point $z_0 = r_0 e^{i\theta_0}$ and suppose that
(i) the first order partial derivatives of the function u and v with respect to r and θ exists everywhere in the neighborhood; (ii) those partial derivatives are continuous at (r_0, θ_0) and satisfy the polar form $ru_r = v_\theta$ and $u_\theta = -rv_r$ of Cauchy-Riemann equations at (r_0, θ_0).Then $f'(z_0)$ exists, its value being $f'(z_0) = e^{-i\theta}(u_r + iv_r)$ where R.H.S. is to be evaluated at (v_0, θ_0)
Examples 1, Page No. 70

10.3 **Analytic Functions** [3 Lectures]

(a) Definition of Analytic functions, Entire function, Singular points, Relations between continuity, differentiability and analyticity.
Examples $1, 2, 3$ on Page No. 75, 76.

(b) Definition of Harmonic function and Harmonic conjugate:
Theorem: If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.
Theorem: A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if, and only if, v is a harmonic conjugate of u.
Examples $3, 4, 5$ on Page No. 80, 81.

(c) Reflection Principle
Theorem (Statement only): Suppose that a function f is analytic in some domain D which contains a segment of the x-axis and whose lower half is the reflection of the upper half with respect to that axis. Then$\overline{f(z)} = f(\overline{z})$ for each point z in the domain if, and only if, $f(x)$ is real for each point x on the segment.
Examples on Page No. 97.

10.4 **Exercises** [3 Lectures]

At least the following problems from the following pages of the book “Complex Variables and Applications”, 8th edition, by James Ward Brown & Ruel V. Churchill should be discussed:
1, 2, 3 on Page No. 55; 1 on Page No. 62 ; 1, 2, 3 on Page No. 71; 1, 2, 4 on Page No. 77; 1, 2 on Page No. 81.

Text/Reference Books:

1. T. M. Apostol; *Mathematical Analysis*, 2nd edition, Narosa Publishing House, India.
2. D.V. Wider; *Advanced Calculus*, 2nd edition, Dover Publications.

3. J. W. Brown and R. V. Churchill; *Complex Variables and its Applications*, 8th edition, McGraw-Hill. Inc.
4. Narayan Prasad Pahari; *A Textbook of Mathematical Analysis*, Sukunda Pustak Bhawan, Kathmandu.
5. S. Ponnusamy; *Foundations of Mathematical Analysis*, Springer.

Examination:
There will be a final examination of 70 marks for the period of three hours. The internal examination of 30 marks will be conducted by the department of mathematics of related campus and the marks will be submitted to Tribhuvan University Office of the Controller of Examination, Balkhu. A candidate must pass the internal and the final examinations separately.
Marks allocation for the internal examination:

- Written examinations: 20 marks (1 hour)
- A student or a group of students with presentation: 5 marks
- Assignments: 5 marks

Guidelines to the question setter:
In the final examination
1. Questions must include every unit.
2. There will be two groups, namely, Group A and Group B.
3. In group A, there must be OR selection for 15 marks questions.
4. In group B , there must be OR selection for 20 marks questions.
5. OR Selection must be given from the same unit.
6. Questions must be creative and should be appropriate to the allocated time.

On the basis of the guidelines mentioned, we enclose one set of model question for Mathematical Analysis (Math 427)

Model Question

Tribhuvan University
Faculty of Humanities and Social Sciences

Bachelor Level/ IV year/Humanities — Full Marks: 70
Mathematics (Math 427) — Pass Marks: 28
Mathematical Analysis — Time: 3 hours

Candidates are required to give their answers in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt **All** the questions.

Group A [$2 \times 15 = 30$]

1. (a) State and prove Lindelöf covering theorem. [1+4]
(b) Define uniform continuity of a function on a set in a metric space. Let $f(x) = x^2$ for all x in $A = (0, 1]$. Prove that $f(x) = x^2$ is uniformly continuous on $A = (0.1]$. [1+4]

OR

Let f be defined on an interval S in $\mathbb{R}$. Assume that f is continuous at a point c in S and let $f(c) \neq 0$. Then, prove that there exists a one-ball $B(c; \delta)$ such that $f(x)$ has same sign as $f(c)$ in $B(c; \delta) \cap S$. [5]
(c) Define total variation of a function on $[a, b]$. Assume that f and g are each of bounded variation on $[a, b]$. Then, prove that the function $f \cdot g$ is also bounded variation on $[a, b]$. [1+4]

2. (a) Define the directional derivative. Show that the existence of finite directional derivative $\boldsymbol{f}'(\boldsymbol{c}; \boldsymbol{u})$ of a function $\boldsymbol{f}$ at $\boldsymbol{c}$ in all the directions given by $\boldsymbol{u}$ may not imply the continuity of the function at $\boldsymbol{c}$. [1+4]

OR

Let S be an open subset of $\mathbb{R}^n$ and let $\boldsymbol{f} : S \to \mathbb{R}^n$ is differentiable at each point of S. Let $\mathbf{x}$ and $\mathbf{y}$ be two points of S such that $L(\mathbf{x}, \mathbf{y}) \subseteq \mathbf{S}$. Then, for every point $\mathbf{a} \in \mathbb{R}^{\mathbf{n}}$, there is a point $\mathbf{z}$ in $L(\mathbf{x}, \mathbf{y})$ such that $\mathbf{a}.\{\boldsymbol{f}(\mathbf{y}) - \boldsymbol{f}(\mathbf{x})\} = \mathbf{a}.\{\boldsymbol{f}'(\mathbf{z})(\mathbf{y} - \mathbf{x})\}$. [5]
(b) State any two major differences between pointwise and uniform convergence of a sequence of functions. Show that the sequence

$$f_n(x) = \frac{nx}{1 + n^2x^2}$$

converges pointwise, but not uniformly on $[0, k]$ for $k > 0$. [1+4]

OR

State and prove Dirichlet's test for the convergence of a series of functions. [1+4]
(c) Define principal argument of a complex number wit an example. Prove that the equation of the circle $|z - z_0| = r$ can be written as $|z|^2 - 2\text{Re}(z\overline{z_0}) + |z_0|^2 = r^2$ [1+4]

Group B [$4 \times 10 = 40$]

3. Define a bounded set and an accumulation point of a set $S \subseteq \mathbb{R}^n$. State Bolzano-Weiestrass theorem on $\mathbb{R}^n$ and illustrate by an example that boundedness is not necessary in order to have an accumulation point for an infinite set. Use Bolzano-Weiestrass theorem to prove Cantor's intersection theorem on $\mathbb{R}^n$. [2+3+5]

OR

Define a metric space. Show that

$$d_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \,\text{and}\, d_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

is defined by

$$d_1(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i \leq n} |x_i - y_i| \text{ and } d_2(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

are the metrices in $\mathbb{R}^n$. Also prove that $d_1(\mathbf{x}, \mathbf{y}) \leq ||\mathbf{x} - \mathbf{y}|| \leq d_2(\mathbf{x}, \mathbf{y})$. [1+ 3+3+3]

4. When is a bounded function f said to be be integrable with respect to α on $[a, b]$? Assume that $\alpha \uparrow$ on $[a, b]$. State Riemann's condition for integrability of f with respect to α on $[a, b]$. Assume that $\alpha \uparrow$ on $[a, b]$. If $f \in \mathcal{R}(\alpha)$ on $[a, b]$, then prove that f satisfies Riemann's conditions with respect to α on $[a, b]$. [2+2+6]

OR

State and prove the First Mean Value Theorem for Riemann-Stieltjes Integral. Use it to prove the Second Mean Value Theorem for Riemann-Stieltjes Integral. [1+4+5]

5. (a) Define an improper integral of the first kind. State and prove Cauchy criterion for convergence. [1+4]
(b) Define the convergence of improper integral of the second kind. Investigate the convergence or divergence of the improper integral $\int_a^b \dfrac{dx}{(x-a)^p}$. [1+4]

6. Define harmonic function. Prove that the real and imaginary parts of an analytic function $f(z) = u(x, y) + iv(x, y)$ satisfy Laplace's equation. Find the harmonic conjugate of the function $u(x, y) = xy$ and corresponding analytic function $u + iv$ in terms of z. [2+3+5]

10 Math 428 Detailed Course of Project Work

TRIBHUVAN UNIVERSITY
Faculty of Humanities and Social Sciences
Micro Syllabus

Course Title: Project Work (Elective)	**Full Marks:** 100
Course No.: Math 428	**Pass Marks:** 40
Level: B.A.	**Year:** IV
Nature of Course: Theory (Project writing)	

Course Objectives: This course is designed for the interested students of the fourth year of Four Years B.A. Program as an elective subject in mathematics. This course offers students to learn basics of mathematical research.
Detailed Syllabus

Benefits and importance of Project Writing
Project writing can be considered as the first step in the world of research.
Here we first summarize the benefits of conducting a research as:

- Person with a research degree gets a better employment, promotion, increase in benefits or even gets a position in various academic institutions and industries.
- They will be able to solve the existing unsolved and challenging problems
- They will acquire more respect and better recognition in their society.
- They will develop curiosity to find new useful things in real world so that a society can be benefitted.

There are numerous benefits of writing a project at the undergraduate level. These various advantages along with their importance are:

- It provides learning experience that enhances their knowledge
- It provides students with an opportunity to synthesize knowledge from various areas
- It enables students to acquire essential skills like collaboration, communication and independent learning
- It prepares students for lifelong learning and the challenges ahead
- It gives them the tolerance for obstacles
- It helps to understand to know how knowledge is constructed
- It builds up self confidence in students
- It helps students to clarify their career path
- It helps to improve the student's technical skill
- It helps to determine the area of interest
- It helps to integrate theory and practice
- It helps to analyze data
- It helps to understand the research process
- It helps to develop the skills for the interpretation of results

Guidelines

It is expected that a standard project in mathematics should be organized, properly documented, carefully edited, logical, imaginative, and accurate. Some suggestive guidelines are listed below.

1. Students are required to identify the mathematical problems of their field of interest in the project work through literature review and the problem should be addressed by them.
2. A student or a group of students can carry out project work only if a faculty agrees to supervise student(s) to carry out research activities.

3. The nature of a project work in mathematics can be theoretical, computational or applied type. In any of the cases, students are supposed to critically review literature of the area and identify the problem specifically.

4. Students are required to prepare a proposal of the project and submit it to the department of mathematics of the related campus within the first three months of the commencement of the fourth year. The general format of the proposal should like this:

   (a) Background/Introduction

   (b) Literature Review

      - Systematic identification, location and analysis of documents containing information related to the research problem
      - Helpful to determine whether the topic or problem is worth studying, or manageable
      - Provide understanding about what has been done and what has to be done
      - Helpful to determine the scope and framework of the study
      - Helpful to define or determine theory and methods of the proposed study
      - Helpful to justify the significance of the proposed study
      - Sources of everything

   (c) Motivation/ Objectives

   (d) Methodology

   (e) Anticipated Outcomes if any

   (f) References

5. Any full time faculty (Professor, Reader, Lecturer, Teaching assistant) of mathematics Of Tribhuvan University can be a supervisor. Part time faculties of TU can serve as a co-supervisor if needed. Any masters holder in mathematics and serving in other areas such as government job, NGO, INGO can also be a co-supervisor. A faculty of mathematics department of TU may supervise maximum of 5 students at a time if it is an individual project work and 10 students if it is a group work.

   Role of supervisor: The faculties who supervise the math projects are equally responsible to complete the project. There are basically three different roles of a supervisor in project writing. He works as resource provider, becomes a co-learner, and also as facilitator and counselor. It is very important that faculties who supervise the math projects have to ensure that students have a complete and deep understanding of the chosen topic of the project. Most importantly the supervisor should have good knowledge of the mathematical subject matter.The various roles of the supervisor are listed as:

   - help students select a mathematical topic
   - help students generate ideas through brainstorming
   - plan the project with the student
   - guide students to formulate their project objectives.
   - help students gather ideas, define objectives, draw up the schedule and provide input for language skills.
   - intervene if student's direction not practical
   - offer suggestions to solve problems
   - center on what students learned during the project
   - share the reflection
   - provide a balanced picture of strengths and weaknesses
   - offer suggestions for improvement
   - read the work at most in chapters
   - use different colored ink to add comments
   - before submission, the recommended corrections must be implemented

6. Students appeared in the final exams of first and second year and have passed at least first or second year are eligible to take the project work.

7. Additional fee for the project and remuneration for the supervisor will be decided by the concerned university campus in coordination with the Dean's office.

Format of Project Work

The various components of a standard project writing are listed as:

1. Title and title page
2. Dedication
3. Copy right
4. Deceleration
5. Certificate of Approval
6. Acknowledge
7. Abstract
8. Table of Contents
9. List of Figure/table/symbols
10. Main body of the project
11. Conclusion
12. References and Bibliography

We now describe these components in detail. We begin with title and title page.

1. **Title and title page**
   Title is very important in project writing, so students and their advisors should give high effort to choose a title. At this level of mathematical writing, it is not expected from the students to obtain a new results or a breakthrough in the assigned topics. They are expected to study and understand the topics in full entirety and look for various applications in various areas. Some of the features of the title and title page are:
   - This page contains short, descriptive title of the proposed project.
   - Title should be fairly self-explanatory.
   - It contains information about the author, institution, department and date of delivery.
   - The main goal is to give clear understanding of the content.
2. **Dedication**
   In this page, we write the various names of the people whom we want to dedicate our work.
3. **Copyright**
   On this page, we write the name who has the copyright of the work.
4. **Deceleration**
   Here, the author declares the whole project is his own work and has not been published anywhere else. Moreover, author also declares that any literature, data or works done by others and cited within this dissertation has been given due acknowledgment and listed in the reference section.
5. **Certificate of Approval**
   In this page, evaluation committee members, supervisor, head of the department, internal and external examiner certify that the project work is satisfactory and meets the requirements and sign the page.
6. **Acknowledge**
   In this page, author gives acknowledgement to all the person who helped him/her directly or indirectly in his/her project. This is an optional page. In general, authors acknowledge their project supervisor, head of the department, teaching faculties, their family members and friends.
7. **Abstract**
   Abstract is very important in project writing and it is generally written at the completion of the entire work. A good reader can easily understand the entire project from its abstract. A good abstract states the brief summary of the project and should be written in simple declarative sentences. It gives a brief summary of how one wants to address the issue and includes a possible implication of the proposed work. One should note that an abstract should not review the report but should act as a sample of the contents of the project. In general its length should not exceed 200 words.

8. **Table of contents**
   Here, we list all headings and subheading with page numbers. This page does not include table of contents and abstract.

9. **List of figure/table/symbols**
   All the figures and tables used in the project are listed with their respective page numbers. Moreover, we also list all symbols that are used in the whole work with their description.

10. **Main body of the project**
    In this section all the work of the project are included. This section of the project generally has three chapters and are listed below.

    (a) **Chapter One**:
    This chapter is also called the introduction chapter. In this chapter we mainly discuss the historical background and objectives of the proposed work. Various motivations of the project, its significance and scope are also discussed. We should also enlist or discuss the possible limitations of the project along with the difficulties during the work.

    (b) **Chapter Two**:
    This chapter is called the preliminary chapter where we give the details of all the mathematical principles behind the entire work relative to the project. The information collected from the literature reviews are kept in this chapter. Here, the author should describe the theoretical/mathematical principles behind the whole work relative to the project. Moreover, all the mathematical definitions, theorems, lemmas and examples which are used in the last chapter are stated. Proofs and details are also presented or referred from some sources. Methodologies of the work are also discussed in this chapter.

    (c) **Chapter Three**:
    This chapter is the main chapter of the project where we discuss the findings of the project. So this is the main body of the work and authors primarily focus on this chapter.

11. **Conclusion**
    In this section we review the entire work and present it in concise form.

12. **References and Bibliography**
    In order to support and justify an idea in a project work, authors generally take the help of books, article, thesis etc. In return, authors have to thank authors of the used source. This is called referencing. Thus, referencing is simply a method of acknowledging and recognizing the authors for their innovative work. Similarly in some of the academic writings, Bibliography is used in the substitution of Reference list. There is a simple difference between bibliography and reference list. In the bibliography we list all the sources (books, articles, thesis, project, web-sites etc) that are referred in the main text and it also includes all the sources consulted even if they are not cited or refereed in the work for the future use. On the other hand, References only include the sources that are cited or referred in the work.
    Now we discuss the various referencing styles.

    Referencing Styles
    There are various standard referencing styles available. In general the institutions or publishing houses decide their referencing style so that the referencing styles vary with publishing houses. The commonly and widely used referencing styles are listed below:

    1. Vancouver
    2. Harvard
    3. APA (American Psychological Association)
    4. MLA (Modern Language Association)
    5. Chicago
    6. ACS (American Chemical Society)
    7. AGLC (Australian Guide to Legal Citation)
    8. AMA (American Medical Association)
    9. CSE (Council of Science Editors)
    10. IEEE (Institute of Electrical and Electronics Engineers)
    11. AMS (American Mathematical Society)

For mathematics, we use AMS style of referencing. We now give the details of referencing in AMS style for various sources. In AMS style of referencing, in-text citation is simply done by the number in the square bracket where the number corresponds to the position it appears in the reference list.

We follow the following sequence to list the article in the reference list.

1. Author's first name and last name
2. Title of article (in italics)
3. Journal Title
4. Volume Number (Year of Publication)
5. Page
6. DOI number (if exists)

We follow the following sequence to list the book, thesis, project in the the reference list.

1. Author's first name and last name
2. Title (in italics)
3. Name of the publisher
4. City of Publication
5. Year of Publication

Using the above style, we now show the listing of an article and a book in the reference list.

1. S. Ghimire and C. N. Moore, *A Lower Bound in the Tail Law of the Iterated Logarithm for Lacunary Trigonometric Series*, Proceeding of American Mathematical Society, 142(9), 3207-3216.
2. James Stewart, *Calculus: Early Transcendental*, USA, Cengage Learning, 2008.

Plagiarism

The word plagiarism is very important term for every researcher and research-oriented person. All of the following actions are considered as plagiarism

- turning in someone else's work as your own
- copying words or ideas from someone else without giving credit
- failing to put a quotation in quotation marks
- giving incorrect information about the source of a quotation
- changing words but copying the sentence structure of a source without giving credit
- copying so many words or ideas from a source that it makes up the majority of your work, whether you give credit or not

Evaluation

The project work submitted to the Mathematics department after the approval of the supervisor will be reviewed. After the approval of a committee, student(s) present the project work.

The evaluation scheme is outlined below.

- Introduction of the Topic: 10%
- Presentation: 15%
- Organization: 10%
- Figures/plots/tables: 5%
- Content: 60%

The breakdown of the marks for the content (60%) is:

1. Originality and Creativity: 15%
2. Literature Review: 10%
3. In-depth Research: 15%
4. Analysis and Logical Argument/Presentation: 15%
5. Conclusion / Findings: 5%

The final presentation/viva examination should be held within a couple of months of the fourth year final examination and the marks will be submitted to Tribhuvan University Office of the Controller of Examination, Balkhu.

Project Work in Mathematics

At the beginning of the project writing in mathematics, a student has to choose an area in which he/she wishes to write a project. The student should be familiar with the various branches of mathematics. The various branches of mathematics are listed in alphabetical order:

- Algebra (Linear, Multilinear, abstract, Elementary)
- Arithmetic (Number Theory)
- Calculus (Analysis, Differential Equations, Dynamical Systems,Numerical Analysis, Optimization, Functional Analysis)
- Geometry (Discrete, Algebraic, Analytic, Differential, finite, topology, trigonometry)
- Foundations (Philosophy of Mathematics, Mathematical Logic, Set theory, Category theory)
- Applied (Mathematical Physics, Probability, Mathematical Statistics, Statistics, Game theory, Information theory, Computer Science, Computational, Control theory)
- History of Mathematics, Mathematics and art
- Mathematics Education, Order theory, Graph theory

After the successful completion of the project writing in mathematics, the various benefits are:

- It increases the mathematical understanding of the students. More precisely, it helps students understand a specific math concept or idea by doing in-depth study of these concepts.
- It develops confidence in students while studying or teaching mathematics.
- They will enjoy doing mathematics.
- They will develop problem solving skills.
- They will develop creative and imaginative approaches to understand mathematics.

The mathematics projects can be written in a number of ways. Some of the ways are:

- To in-depth study of a widely used theorem and its consequences.
- To solve a real life problem.
- To solve a problem arising in other disciplines.
- To study how a concept is used to create a new knowledge.
- To study the history of various mathematicians.

A few possible topics:

- First order differential equations and its various applications in real world.
- The Relevance of the use of the method of undetermined coefficients for solving differential equations
- The application of linear programming in profit maximization

- Studying different numerical methods in solving first order differential equations
- Numerical methods for solving partial differential equations
- Modeling the effects of carriers on transmission dynamics of Infectious diseases
- An Exploration and Analysis of Mathematics as a tool in the Arts
- Linear and non-linear sampling theory with applications to current problems in communications
- Use of Calculus in Business
- Contribution of Leibnitz in Mathematics
- Fibonacci Number in Nature
- Golden Ratio
- Laplace transform and its applications
- Applications of Z-transform in Engineering
- A study on Internet Mathematics
- Mathematics and Sound
- Random Walks
- The development of matrix theory
- Ito's integral
- Stochastic Processes and their Applications to Mathematical Finance
- How is Lagrange's theorem related to RSA cryptography?
- The application of fractional calculus in Mathematics and Physics
- Statistical modeling for data that are dependent in space and/or time.
- How was algebra invented?
- Relationships between theorems of linear algebra and matrix theory
- A study on the Lotka-Volterra Equations and the Undamped Pendulum
- Numerical Error in Euler's Method
- Qualitative behavior of systems with eigenvalues
- Qualitative Analysis of Autonomous Systems
- Population Dynamics
- Hamiltonian and Dissipative systems
- Discrete Logistic Equation
- Mathematics behind Bungy jumping

References:

1. Alvi, Mohsin., *A Manual for Referencing Styles in Research*, https://www.researchgate.net/publication/308786787, 2016.
2. C.R. Kothari: *Research Methodology; Methods and Techniques*, Second Edition, New Age International Publishers.
3. Durga Jang K.C. and Santosh Ghimire, Basics and Structure of Project Writing in mathematics, Journal of Nepal Mathematical Society(JNMS), Volume 3, Issue 1 (2020), 21-30.
4. S.B. Mishra and S. Alok: *Handbook of Research Methodology*, Educreation Publishing.

11 Math 429 Detailed Course of Mathematical Economics

TRIBHUVAN UNIVERSITY
Faculty of Humanities and Social Sciences
Micro Syllabus

Course Title: Mathematical Economics (Elective) **Full Marks:** 100
Course No.: Math 429 **Pass Mark:** 40
Level: B.A. **Year:** IV
Nature of Course: Theory **Periods:** 9 Lecture hours/Week

Course Objectives: This course is designed for the interested students of the fourth year of Four Years B.A. Program as an elective subject in mathematics. It aims to introduce mathematical modes of real-life problems related to economics that exist in the society and industry. After the completion of this course, the students will be able to understand mathematical modeling techniques of different economic problems and apply mathematical tools to solve them. The focus has been given to the equilibrium, cooperative-static and dynamic analysis, and optimization techniques.

Course Contents:

Unit 1. Economic Models and Equilibrium Analysis [18 Lecture Hrs]

Nature of mathematical economics, Economic models, Equilibrium analysis, Market equilibrium, Linear, non-linear and general models, Solution by elimination of variables, algebraic method and graphical method, General equation system, Applications to national-income analysis, Finite Markov chains, Applications to market and income models, Limitations of static analysis.

Details.

1. Meanings of mathematical and non mathematical economics, econometrics, variables, constants and parameters for modeling in mathematical economics, equations and identity, relations and functions in economic models, representation of cost as a function of output, demand curve, average fixed curve, production function, system of equations in mathematical economic models.
2. Meaning of equilibrium, partial market equilibrium, a linear model and its graphical interpretation, solution by elimination of variables, a nonlinear model, quadratic equation and quadratic functions with graphical interpretation, graphical solution, general market equilibrium, two-commodity market model with illustrative example (Book 1, page 42), n-commodity model and solution to general-equation system for economic models.
3. Equilibrium in national income analysis, the mathematical model, interpretation of the parameters and variables involved in the model, total purchase coast as as an inner product of two vectors (Book 1, Example 7, page 54), matrix representation of simple national income model (Book 1, Example 11, page 55).
4. Finite Markov chains with special case, the market and national income models, the IS-LM model, Leontief input-output models, limitations of static analysis.
5. Related problems supporting the definitions, theories, economic models and solution methods.

Unit 2. Application of Derivatives in Economics [13 Lecture Hrs]

Concept and use of derivatives in comparative statics, Modeling of marginal and average revenue functions, Relationships between the cost functions, Gradient vector of the production function, Applications to comparative static analysis, Geometric interpretations in economic terms.

Details.

1. Nature of comparative statics, derivative and slope of a total-cost function, the marginal cost.
2. Finding marginal revenue function, form average revenue function, geometry of marginal revenue curve, relation between marginal and average cost functions, graphical representation.
3. The production function, geometrical interpretation of partial derivatives, gradient vector of the production function.
4. Application to comparative static analysis, the market model, geometry, national income model, the input-output model.
5. Related problems with economic terms.

Unit 3. Total Differentials and Implicit Functions [13 Lecture Hrs]

Meaning of total differentials and implicit functions in economic terms, Comparative statics of general function models, National income model and its extension, Application to economical problems, Limitations of comparative statics.

Details.

1. Differentials and its geometrical interpretation, definition of point elasticity and application of differential in economic terms, Example 1, Example 2 (Book 1, page 182) and the corresponding geometry.
2. The total differentials of a saving function, functions of n independent variables, rules of differentials, total derivative and its variations, Example 2, Example 3 (Book 1, page 191).
3. Implicit functions and the derivatives, illustration with economic functions (Example 4, Book 1, page 198-199), extension to the simultaneous equations case, Example 6 (Book 1, page 203-204).
4. The market model, simultaneous-equation approach, use of total derivatives.
5. The national income IS-LM model, their curves and slopes, and extension of the model, limitation for comparative statics.
6. Applications of the models with illustrative examples in economic terms.

Unit 4. Optimality Conditions [13 Lecture Hrs]

Equilibrium analysis, Meaning of optimum and extreme values, Relative and absolute optimality, First, second and higher derivative tests, Necessary and sufficient conditions, Conditions for profit maximization, Marginal revenue curve, Geometric interpretations.

Details.

1. Meaning of equilibrium in economics, goal of equilibrium, meaning of minimum, maximum and optimum, criterion of optimization problems, the objective function and the choice of variables and objective that maximize the difference between revenue and cost functions, meaning of relative maximum and minimum, relative versus absolute extrema, graphical representations.
2. First-derivative test, Example 2 (Book 1, page 226), second and higher derivatives, derivative of a derivative, interpretation of the second derivative, convex concave functions, application to quadratic functions.
3. The second derivative test, second derivative test for relative extremum,necessary and sufficient conditions, conditions for profit maximization, its geometry, Example 3 (Book 1, page 238), the marginal revenue function, their geometrical interpretations.
4. The Taylor series expansion and extremum of a function, statements of n-th derivative test conditions.
5. Test of the above criterion with economic functions as problems.

Unit 5. Exponential and Logarithmic Functions [13 Lecture Hrs]

Economic applications, Computation of interest, The growth function and its variants, Optimal timing.

Details.

1. Definition, nature, meaning and graphical illustrations of exponential and logarithmic functions, illustrative examples.
2. Problem of growth, economic interpretation of e, interest computing and the function Ae^{rt}, instantaneous rate of group, continuous versus discrete growth.
3. Optimal timing, a problem of wine storage, maximization conditions, graphical representation, timber cutting problem.

Unit 6. Optimization Methods [18 Lecture Hrs]

Differential versions, First and second order conditions, Extreme values of functions of two variables, First and second order conditions, Quadratic forms and objective functions with more than two variables, Conditions to convexity and concavity, Their geometry.

Details.

1. Objective function with more than one variable, differential version of optimization conditions, first-order condition, second-order condition, differential versus derivative conditions, their geometry.
2. Extreme values of functions of two variables, first-order condition, second-order partial derivatives, second-order total differential, second-order condition, their geometry.

3. Objective functions and quadratic forms with two variables, second-order total differential as a quadratic form, positive and negative definiteness, determinantal test for sign definiteness, Jacobian and Hessians, ideas of extensions in case of more than three variables.
4. Convex functions and convex sets, second-order conditions in relation to convexity and concavity, test of convexity and concavity, differentiable functions.
5. Geometrical representations of the above concepts supported with relative problems (with economic meanings as far as possible).

Unit 7. Constrained Optimization [18 Lecture Hrs]
Optimization with equality constraints, Effects of a constraint, Stationary values, Lagrange-multiplier method and an interpretation, Total differential approach, Second order total differentials, Second order conditions, The Bordered Hessian, multi-variable and multi-constraints.

Details.

1. Meaning of constraint optimization, free optimum and constrained optimum, effects of a constraint, find the stationary values, Lagrange multiplier, Lagrange functions, method of Lagrange-multiplier, the methods illustrated bu suitable examples (e.g., Book 1, Example 1 and Example 2).
2. Total differential approach, an interpretation of the Lagrange multiplier.
3. Second-order total differential, second order conditions, the Bordered Hessian, illustrated by examples (e.g., Book 1, pages 359-361).
4. Multi-variable and multi-constraints (only models).

Unit 8. Economic Applications [13 Lecture Hrs]
Utility maximization and consumer demand, First order and second order conditions, Homogeneous functions, Linear homogeneity, Cobb-Douglas production function, Economic interpretations.

Details.

1. Definition of utility function, first-order condition, geometry, second order condition.
2. Homogeneous functions, linear homogeneity, production function, proprieties (Book 1, pages 384-386).
3. Cobb-Douglas production function and extension of the results.
4. Related problems (mostly with respect to economic function).

Unit 9. Nonlinear Programming [18 Lecture Hrs]
Effect of non-negativity restrictions and inequality constraints, Nonlinear utility function maximization, KKT-conditions, The inequalities at boundary points, Meaning of constraint qualifications, Sufficient conditions to NLP.

Details.

1. Definition of nonlinear programming, type of restrictions, effect of non-negativity restrictions, effect of inequality constraints, nonlinear utility function maximization, illustrations by examples (e.g., Example 1, Book 1, page 406).
2. Kuhn-Tucker conditions, n-variable, m-constraint cases (only idea), the Kuhn-Tucker conditions illustrated by examples (e.g. Example 1, Book 1, page 410), the case of minimization function.
3. The inequalities at boundary points, idea on the constraint qualifications (supported by examples and graphs, e.g., Book 1, pages 412-416), Kuhn-Tucker sufficiency theorem for concave programming (Book 1, page 424-425).
4. Related problems (preferably with economic functions).

Unit 10. Dynamic Analysis [13 Lecture Hrs]
Dynamics and integration, The growth model, Dynamics of market price, The qualitative-graphic approach, The time path, The market model with price expectations.

Details.

1. Meaning of dynamic analysis, population growth problem formulations (14.1 and 14.2, Book 1, page 445), initial and boundary conditions, method of integration.
2. Some economic applications of integrals, from a marginal function to a total function, investment and capital formulation problem (14.10, Book 1, page 466), the Domar growth model framework and the solution.

3. Dynamics of market price, the framework, time path.
4. The qualitative-graphic approach, the phase diagram, types of time path.
5. The market model with price expectations, price trend and price expectations, a simplified model, time path of price. 1
6. Related problems.

Text/Reference Books:

1. A.C. Chiang, K. Wainwright: *Fundamental Methods of Mathematical Economics*, McGraw Hill Publishers.
2. M. Hoy, J. Livernois, C. McKenna, R. Rees, T. Stengos: *Mathematics for Economics*, Third edition, PHI Learning Private Limited.

Examination:

There will be a final examination of 70 marks for the period of three hours. The internal examination of 30 marks will be conducted by the department of mathematics of related campus and the marks will be submitted to Tribhuvan University Office of the Controller of Examination, Balkhu. A candidate must pass the internal and the final examinations separately.

Marks allocation for the internal examination:

- Written examinations: 20 marks (1 hour)
- A student or a group of students with presentation: 5 marks
- Assignments: 5 marks

Guidelines to the question setter:

In the final examination
1. Questions must include every unit.
2. There will be two groups, namely, Group A and Group B.
3. In group A, there must be OR selection for 15 marks questions.
4. In group B , there must be OR selection for 20 marks questions.
5. OR Selection must be given from the same unit.
6. Questions must be creative and should be appropriate to the allocated time.

On the basis of the guidelines mentioned, we enclose one set of model question for Mathematical Economics (Math 429)

Model Question

Tribhuvan University
Faculty of Humanities and Social Sciences

Bachelor Level/ IV year/Humanities | Full Marks: 70
Mathematics (Math 429) | Pass Marks: 28
Mathematical Economics | Time: 3 hours

Candidates are required to give their answers in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt **All** the questions.

Group A [$2 \times 15 = 30$]

1(a). How do you distinguish mathematical economics and econometrics? Distinguish parameters and variables in modeling them. Construct the linear partial-equilibrium market model for price determination in an isolated market. Explain the model with graphical interpretation. [1+1+3+3]

(b). Given an example of a quadratic cost as a function of output. Consider the following closed economy IS-LM model

$$\begin{aligned} Y &= C + I + G \\ C &= a + b(1-t)Y \\ I &= d - ei \\ G &= G_0 \end{aligned}$$

where the symbols have their usual meanings. Write the system in matrix form and discuss its solution procedure. Whats are the limitations of static analysis? [5+2]

OR

Define a convex function with a suitable economic example and a graphical representation. Is a relative minimum point of a strictly convex function also a unique absolute minimum? Check the convexity of the function $z = x_1^2 + x_2^2$. [3+1+3]

2(a). Write nonlinear programming model with inequality constraints. Discuss the effect of these constraints. [2+6]

OR

Write KKT conditions for the nonlinear programming with inequality restrictions. Solve the problem max $z = xy$ subject to $x + y \leq 100,\ x \leq 40,\ x, y \geq 0$. [2+6]

(b). How do you relate economic dynamics in terms of integral calculus? Give the framework of Domar growth model and solve it. [1+3+3]

Group B [$4 \times 10 = 40$]

3. Given the total cost function $C = f(Q)$, where C denotes the total cost and Q the output, define the marginal cost (MC) and explain their relation in terms of derivative. Draw the cost curve and tangent line to it, and explain the geometry in detail when the cost vary. By considering an example, $C = Q^3 - 4Q^2 + 10Q + 75$, show that the fixed cost cannot affect the marginal cost. Illustrate in graph a primitive total function and a marginal function in economic terms. [2+3+2+3]

4. Consider one-commodity market model and use rules of differentiation to explain how the equilibrium value of an endogenous variable will change with respect to a change in any of the exogenous variables? [10]

OR

Write a general one-commodity market model where the quantity demanded is a function of price and income. Discuss simultaneous-equation model to analyze it. How are the total derivatives used to analyze it. [2+5+3]

5(a). Express the profit function $\pi(Q) = R(Q) - C(Q)$ in terms of total revenue and total-cost function and text necessary and sufficient conditions for profit maximization. Represent your results graphically. [3+2] (b). What are economic applications of exponential and logarithmic functions? Consider a wine store maximization problem $A(t) = Ke^{\sqrt{t}-rt)}$, where the symbols has their usual meanings. Determine the range value of t so that the profit is maximized. [1+4]

6(a) Consider a function $z = f(x, y)$ of two variables. Discuss the use of total differentials for the first order and second order condition of extreme points for this function? What are these conditions if the function is of single variable? Find the extreme values of $z = x^2 + 2y^2 + xy + 3$. [2+1+2]

OR

Write second order necessary and sufficient conditions for the optimization problem $z = f(x, y)$ subject to the equality constraint $g(x, y) = c$. Determine whether $z = 4x^2 + 4xy + 3y^2$ subject to $x - 2y = 0$ is either positive definite. [1+4]

(b). For optimizing a function $z = f(x, y)$ subject to the equality constraint $g(x, y) = c$, write the general form of Lagrangian function. Use this method to solve the problem: $\max z = xy + 2x$ subject to the constraint $4x + 2y = 60$. [1+4]

OR

What do you understand by linear homogeneity property? Using the linear homogeneity property for the production function $Q = f(Q, L)$, prove that the average physical of labor and of capital can be expressed as functions of capital-labor ratio. Also prove that the Eulor's theorem holds for this function. [1+2+2]