SOME CONSTRUCTIONS OF GORENSTEIN ARTIN RINGS

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Abstract. Using inverse systems, one can construct Gorenstein Artin rings, including the idealisation of a canonical module of an Artinian local ring, as seen in the preparatory talk. Given two Gorenstein Artin rings, we use inverse systems to construct a new Gorenstein Artin ring called their connected sum. Finally, we see a condition under which the idealisation is a connected sum.

1. Introduction

Consider the polynomial rings $T_1 = \mathbb{C}[y_1, \ldots, y_m]$, $T_2 = \mathbb{C}[z_1, \ldots, z_n]$, and let $E_1 = \mathbb{C}[Y_1, \ldots, Y_m]$, $E_2 = \mathbb{C}[Z_1, \ldots, Z_n]$. Then T_i acts on E_i by partial differentiation.

Let $F_1 \in E_1$, $F_2 \in E_2$. Recall that $R_i = T_i/\operatorname{ann}_{T_i}(F_i)$ are Gorenstein Artin, for $i = 1, 2$.

Define $F = F_1 + F_2 \in E = \mathbb{C}[Y_1, \ldots, Y_m, Z_1, \ldots, Z_n]$. Let $T = \mathbb{C}[y_1, \ldots, y_m, z_1, \ldots, z_n]$, and $R = T/\operatorname{ann}_T(F)$. Then R is Gorenstein Artin, and is called a connected sum of R_1 and R_2 over $\mathbb{C}$, denoted $R_1\#_{\mathbb{C}}R_2$.

A natural question is:

Question 1.1. *How are the defining ideals of R, R_1 and R_2 related?*

This is answered in Remark **??**. We first introduce some notation.

Note. The material in the rest of Section 1, and in Sections 2 and 3 is taken from [**?**]. Some of the material can also be found in [**?**] and [**?**, Chapter 4].

Notation.

a) Let $(T, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring. The *socle* of T is $\operatorname{soc}(T) = \operatorname{ann}_T(\mathfrak{m})$. The *Loewy length* of T is $\ell\ell(T) = \max\{n : \mathfrak{m}^n \neq 0\}$.[1]

   Observe that T is not a field if and only if $\ell\ell(T) \geq 1$. Furthermore, if T is Gorenstein, then $\operatorname{soc}(T) \subset \mathfrak{m}^2$ if and only if $\ell\ell(T) \geq 2$.

b) For positive integers m and n, $\mathbf{Y}$ and $\mathbf{Z}$ denote the sets of indeterminates $\{Y_1, \ldots, Y_m\}$ and $\{Z_1, \ldots, Z_n\}$ respectively, and $\mathbf{Y} \cdot \mathbf{Z}$ denotes $\{Y_iZ_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

c) For an ideal J of $\mathsf{k}[\mathbf{Y}]$ or $\mathsf{k}[\mathbf{Z}]$, J^e denotes its extension to $\mathsf{k}[\mathbf{Y}, \mathbf{Z}]$ via the natural inclusions $\iota_Y : \mathsf{k}[\mathbf{Y}] \hookrightarrow \mathsf{k}[\mathbf{Y}, \mathbf{Z}]$ and $\iota_Z : \mathsf{k}[\mathbf{Z}] \hookrightarrow \mathsf{k}[\mathbf{Y}, \mathbf{Z}]$ respectively.

In the next two sections, we see the definition and some basic properties of fibre products and connected sums. When $(R, \mathfrak{m}_R, \mathsf{k})$ and $(S, \mathfrak{m}_S, \mathsf{k})$ are equicharacteristic, i.e., they contain a field and hence their common residue field k, we can say more about their fibre products and connected sums over k. These are recorded in Remarks **??** and **??**.

[1] If T is Gorenstein Artin, its Loewy length is also referred to as *socle degree* in the literature.

2. Fibre Products

Definition 2.1. *Let $(R, \mathfrak{m}_R, \mathsf{k})$ and $(S, \mathfrak{m}_S, \mathsf{k})$ be local rings. The fibre product of R and S over k is the ring $R \times_{\mathsf{k}} S = \{(r, s) \in R \times S : \pi_R(r) = \pi_S(s)\}$, where π_R and π_S are the natural projections from R and S respectively onto k.*

Remark 2.2. With the notation as in Definition **??**, set $P = R \times_{\mathsf{k}} S$. Then, by identifying $\mathfrak{m}_R$ with $\{(r, 0) : r \in \mathfrak{m}_R\}$ and $\mathfrak{m}_S$ with $\{(0, s) : s \in \mathfrak{m}_S\}$, we see that:

a) P is a local ring with maximal ideal $\mathfrak{m}_P = \mathfrak{m}_R \times \mathfrak{m}_S$.
b) (Trivial fibre product) Every ring is trivially a fibre product over its residue field. Indeed, if $(P, \mathfrak{m}_P, \mathsf{k})$ is a local ring, then $P \times_{\mathsf{k}} \mathsf{k} \simeq P$.
c) It follows from [**?**, (1.0.3)] that a local ring $(P, \mathfrak{m}_P, \mathsf{k})$ decomposes nontrivially as a fibre product over k if and only if $\mathfrak{m}_P = \langle y_1, \ldots, y_m, z_1, \ldots, z_n\rangle$ for $m, n \geq 1$ with $\langle \mathbf{y}\rangle \cap \langle \mathbf{z}\rangle = 0$. In this case, we can write $P \simeq R \times_{\mathsf{k}} S$, where $R = P/\langle \mathbf{z}\rangle$ and $S = P/\langle \mathbf{y}\rangle$.

Remark 2.3 (Fibre Products in Equicharacteristic)**.**
Let $R = \mathsf{k}[Y_1, \ldots, Y_m]/I_R$ and $S = \mathsf{k}[Z_1, \ldots, Z_n]/I_S$ be k-algebras with $I_R \subseteq \langle \mathbf{Y}\rangle^2$ and $I_S \subseteq \langle \mathbf{Z}\rangle^2$. Let $P \simeq R \times_{\mathsf{k}} S$. Then $P = \mathsf{k}[\mathbf{Y}, \mathbf{Z}]/I_P$ where

a) $I_P = I_R^e + I_S^e + \langle \mathbf{Y} \cdot \mathbf{Z}\rangle$, $\langle \mathbf{Y}\rangle \cap \langle \mathbf{Z}\rangle \subseteq I_P \subseteq \langle \mathbf{Y}, \mathbf{Z}\rangle^2$, and
b) $I_R = I_P \cap \mathsf{k}[\mathbf{Y}]$, and $I_S = I_P \cap \mathsf{k}[\mathbf{Z}]$.

3. Connected Sums

If $(R, \mathfrak{m}_R, \mathsf{k})$ and $(S, \mathfrak{m}_S, \mathsf{k})$ are Artinian local rings, neither of which is a field, then $R \times_{\mathsf{k}} S$ is not Gorenstein. A *connected sum* of R and S is a quotient of $R \times_{\mathsf{k}} S$ that is Gorenstein.

Definition 3.1. *Let $(R, \mathfrak{m}_R, \mathsf{k})$ and $(S, \mathfrak{m}_S, \mathsf{k})$ be Gorenstein Artin local rings different from k. Let $\operatorname{soc}(R) = \langle \delta_R\rangle$, $\operatorname{soc}(S) = \langle \delta_S\rangle$. Identifying δ_R with $(\delta_R, 0)$ and δ_S with $(0, \delta_S)$, a connected sum of R and S over k, denoted $R\#_{\mathsf{k}}S$, is the ring $R\#_{\mathsf{k}}S = (R \times_{\mathsf{k}} S)/\langle \delta_R - \delta_S\rangle$.*

Connected sums of R and S over k depend on the generators of the socle δ_R and δ_S chosen. For example, the connected sums $Q_1 = (R \times_{\mathsf{k}} S)/\langle y^2 - z^2\rangle$ and $Q_2 = (R \times_{\mathsf{k}} S)/\langle y^2 - 5z^2\rangle$ of $R = \mathbb{Q}[Y]/\langle Y^3\rangle$ and $S = \mathbb{Q}[Z]/\langle Z^3\rangle$ are not isomorphic as rings, as shown in [**?**, 3.1].

Remark 3.2. With notation as in Definition **??**, set $P = R \times_{\mathsf{k}} S$ and let $Q = R\#_{\mathsf{k}}S$.

a) (Trivial connected sum) Every Gorenstein Artin local ring that is not a field is trivially a connected sum over its residue field. In order to see this, consider $(R, \mathfrak{m}_R, \mathsf{k})$ to be a Gorenstein Artin local ring with $\ell\ell(R) \geq 1$ and S be a k-algebra of length two. Note that this forces S to be Gorenstein. One can check that $R\#_{\mathsf{k}}S \simeq R$.
b) If $\ell\ell(R), \ell\ell(S) \geq 1$, then Q is a Gorenstein Artin local ring with $\ell\ell(Q) = \max\{\ell\ell(R), \ell\ell(S)\}$. This is proved in [**?**, 4.4]; see [**?**, 2.8] for a more general result.

We can now answer Question **??**.

Remark 3.3.
Let $R = \mathsf{k}[\mathbf{Y}]/I_R$, $S = \mathsf{k}[\mathbf{Z}]/I_S$ and $P = R \times_{\mathsf{k}} S \simeq \mathsf{k}[\mathbf{Y}, \mathbf{Z}]/I_P$ be as in Remark **??**.

Suppose R and S are Gorenstein and $Q = R\#_{\mathsf{k}}S$ is their connected sum over k. Then, by Definition **??**, there exist $\Delta_R \in \mathsf{k}[\mathbf{Y}]$ and $\Delta_S \in \mathsf{k}[\mathbf{Z}]$ such that their respective images $\delta_R \in R$ and $\delta_S \in S$ generate the respective socles and $Q \simeq (R \times_{\mathsf{k}} S)/\langle \delta_R - \delta_S\rangle$.

Thus $Q \simeq \mathsf{k}[\mathbf{Y}, \mathbf{Z}]/I_Q$, where $I_Q = I_P + \langle \Delta_R - \Delta_S\rangle = I_R^e + I_S^e + \langle \mathbf{Y} \cdot \mathbf{Z}\rangle + \langle \Delta_R - \Delta_S\rangle$.

Question 3.4. When is a Gorenstein Artin local ring decomposable as a connected sum over its residue field?

The following characterisation of connected sums of k-algebras is proved in [**?**].

Theorem 3.5 (Connected Sums in Equicharacteristic)**.** *Let Q be a Gorenstein Artin local k-algebra with $\ell\ell(Q) \geq 2$. Then the following are equivalent:*

i) *$Q \simeq R\#_{\mathsf{k}}S$ for Gorenstein Artin k-algebras R and S with $\ell\ell(R)$, $\ell\ell(S) \geq 2$.*
ii) *$Q \simeq \mathsf{k}[Y_1, \ldots, Y_m, Z_1 \ldots, Z_n]/I_Q$ for m, $n \geq 1$ with $\mathbf{Y} \cdot \mathbf{Z} \subset I_Q \subset \langle \mathbf{Y}, \mathbf{Z}\rangle^2$.*

If the above conditions are satisfied, then we can write $R = \mathsf{k}[\mathbf{Y}]/I_R$, $S = \mathsf{k}[\mathbf{Z}]/I_S$ with $I_R = I_Q \cap \mathsf{k}[\mathbf{Y}]$, and $I_S = I_Q \cap \mathsf{k}[\mathbf{Z}]$.

4. Idealisation of a Fibre Product

Let $R = \mathsf{k}[\mathbf{Y}]/I_R$, $S = \mathsf{k}[\mathbf{Z}]/I_S$ and $P = R \times_{\mathsf{k}} S \simeq \mathsf{k}[\mathbf{Y}, \mathbf{Z}]/I_P$ be as in Remark **??**. Let ω_R, ω_S, and ω_P be respective canonical modules for R, S and P. Recall that the idealisations $Q = P \ltimes \omega_P$, $R' = R \ltimes \omega_R$ and $S' = S \ltimes \omega_S$ are Gorenstein Artin. Since $P = R \times_{\mathsf{k}} S$, a natural question is: Is Q related to R' and S'? If so, what is the relation?

As an answer, we have the following theorem:

Theorem 4.1. *Let $(P, \mathfrak{m}_P, \mathsf{k})$ be an Artinian local k-algebra with a canonical module ω_P. Let R and S be local k-algebras with non-zero maximal ideals, and respective canonical modules ω_R and ω_S. If $P = R \times_{\mathsf{k}} S$, then $Q \simeq R'\#_{\mathsf{k}}S'$, where $R' = R \ltimes \omega_R$ and $S' = S \ltimes \omega_S$.*

This is an application of Theorem **??**. When $\mathsf{k} = \mathbb{C}$, one can use inverse systems to prove it. The key points are the following observations in terms of inverse systems:
i) Let $P = R \times_{\mathsf{k}} S$. If $\omega_R = \langle F_1, \ldots, F_r\rangle \subset \mathbb{C}[\mathbf{Y}]$, and $\omega_S = \langle G_1, \ldots, G_s\rangle \subset \mathbb{C}[\mathbf{Z}]$, then $\omega_P = \langle F_1, \ldots, F_r, G_1, \ldots, G_s\rangle \subset \mathbb{C}[\mathbf{Y}, \mathbf{Z}]$.
ii) If $\omega_{R'} = \langle F\rangle \subset \mathbb{C}[\mathbf{Y}]$, $\omega_{S'} = \langle G\rangle \subset \mathbb{C}[\mathbf{Z}]$, and $Q \simeq R'\#_{\mathsf{k}}S'$, then $\omega_Q = \langle F + cG\rangle \subset \mathbb{C}[\mathbf{Y}, \mathbf{Z}]$, for some $c \in \mathbb{C} \setminus \{0\}$.

We illustrate the proof when $\mathsf{k} = \mathbb{C}$ using the following example:

Example 4.2. Let $R = \mathbb{C}[y_1]/\langle y_1^3\rangle$, $S = \mathbb{C}[y_2]/\langle y_2^4\rangle$. In terms of inverse systems, we can write $\omega_R = \langle Y_1^2\rangle$ and $\omega_S = \langle Y_2^3\rangle$. Furthermore, if $R' = R \ltimes \omega_R$ and $S' = S \ltimes \omega_S$, then R' and S' correspond to $F_1 = Z_1Y_1^2 \in \mathbb{C}[Y_1, Z_1]$, and $F_2 = Z_2Y_2^3 \in \mathbb{C}[Y_2, Z_2]$ respectively.

Let $T = \mathbb{C}[y_1, y_2]$, and $E = \mathbb{C}[Y_1, Y_2]$. Then $P = R \times_{\mathsf{k}} S \simeq \mathbb{C}[y_1, y_2]/\langle y_1^3, y_1y_2, y_2^4\rangle$, and it can easily be checked that $\omega_P = \operatorname{ann}_E(\langle y_1^3, y_1y_2, y_2^4\rangle) = \langle Y_1^2, Y_2^3\rangle$.

Thus, $Q = T'/\operatorname{ann}_{T'}(F)$, where $T' = \mathbb{C}[y_1, y_2, z_1, z_2]$, and $F = Z_1Y_1^2 + Z_2Y_2^3 \in E' = \mathbb{C}[Y_1, Y_2, Z_1, Z_2]$. Since $F = F_1 + F_2$, we see that $Q \simeq R\#_{\mathsf{k}}S$.

References

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