INVERSE SYSTEMS AND IDEALISATION

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Abstract. We start with a ring without a multiplicative identity, and construct a ring with one. Given a ring R and an R-module M, this construction can be mimicked to create a new ring in which M is an ideal, called its idealisation. Using the method of inverse systems, we construct Artinian rings, and special modules associated to them. We then identify the idealisation of these modules in terms of inverse systems.

1. Idealisation

Recall that a ring R is an abelian group under addition with a compatible multiplicative structure. Thus, $(R, +)$ is an abelian group, and we have a multiplication $* : R \times R \to R$ which is associative, i.e., for all a, b, $c \in R$, $a * (b * c) = (a * b) * c$, and distributes over $+$, i.e., for all a, b, $c \in R$, $a * (b + c) = (a * b) + (a * c)$ and $(a + b) * c = (a * c) + (b * c)$.

Notation. Let R be a ring, $n \in \mathbb{Z}$, and $a \in R$. We use the following notation:

$$na = \begin{cases} a + \cdots + a \quad (n \text{ times}) & n > 0 \\ (-a) + \cdots + (-a) \quad (-n \text{ times}) & n < 0 \\ 0 & n = 0 \end{cases} .$$

We say R is a *ring with unity* if it has a multiplicative identity, i.e., there is an element $1 \in R$ such that $a * 1 = a = 1 * a$ for all $a \in R$. If R does not have a multiplicative identity, then following Jacobson ([**?**]), we call R a '*rng*'.

The existence of a multiplicative identity in R leads to good properties, for example, the existence of maximal ideals. But the definition of a ring does not force R to have one. However, we can assume that R contains a multiplicative identity by the following construction:

Construction 1.1. ([**?**]) Let $(R, +_R, *_R)$ be a 'rng'. Set $S = \mathbb{Z} \oplus R$. Define addition and multiplication on S as follows: For m, $n \in \mathbb{Z}$, and a, $b \in R$,
define $(m, a) +_S (n, b) = (m + n, a +_R b)$, and $(m, a) *_S (n, b) = (mn, mb +_R na +_R a *_R b)$.

Remark 1.2.
a) $(S, +_S, *_S)$ is a ring with multiplicative identity $(1, 0)$.
b) The function from R to S given by $a \mapsto (0, a)$ is one-one, and hence R can be identified with the ideal $\{(0, a) | a \in R\}$ of S.
c) S is commutative if and only if R is so.

Observe that
i) $(R, +_R)$ is an abelian group, with a compatible multiplication $\mathbb{Z} \times R \to R$ given by $(n, a) \mapsto na$.
ii) $*_R$ is a multiplication on R which is associative and distributes over addition.

We have used the above properties to create a ring with a unity, in which R is an ideal. This process is called the *idealisation* of R. We now use observations (i) and (ii) to generalise the construction of idealisation to the following setup:

Let R be a ring with 1, and M an R-module. Thus $(M, +_M)$ is an abelian group, with a *compatible 'scalar' multiplication* $R \times M \to M$. For example, if R is a field, then M is a vector space over R.

Note that every R-module M has a multiplication $*_M : M \times M \to M$ given by $(x, y) \mapsto 0$, which is compatible with the addition on M. This is called the *trivial multiplication* on M.

Thus, we can now mimic Construction 1.2, using the pair (R, M) in place of $(\mathbb{Z}, R)$. This gives us the *idealisation* of M, denoted $R \ltimes M$.

Construction 1.3. Let $S = R \oplus M$. For $a,\ b \in R$, and $x,\ y \in M$, define addition and multiplication on S as follows: $(a, x)+(b, y) = (a+b, x+y)$ and $(a, x)\cdot(b, y) = (ab, ay+bx)$.

Remark 1.4.
a) S is a ring with multiplicative identity $(1, 0)$ with operations as above. Furthermore, S is commutative if and only if R is so.
b) The function from M to S given by $x \mapsto (0, x)$ is one-one, and hence M can be identified with the ideal $\{(0, x)|x \in M\}$ of S. Moreover, $M^2 = 0$ in S.

2. Inverse Systems

Note. The material in Sections 2 and 3 is taken from [**?**].

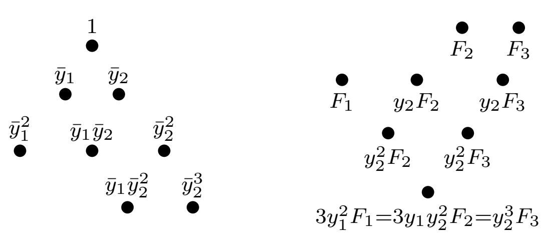
Consider the polynomial ring $T = \mathbb{C}[y_1, \ldots, y_d]$, and let $E = \mathbb{C}[Y_1, \ldots, Y_d]$. Then T acts on E by partial differentiation, i.e.,

for $f \in T$ and $F \in E$, define $f(y_1, \ldots, y_d) \cdot F(Y_1, \ldots, Y_d) = f\left(\frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_d}\right) F(Y_1, \ldots, Y_d)$.

Example 2.1. Let $T = \mathbb{C}[y_1, y_2]$, $E = \mathbb{C}[Y_1, Y_2]$, and $F_1 = Y_1^2$, $F_2 = Y_1Y_2^2$, $F_3 = Y_2^3 \in E$. For $f = y_2^3 \in T$, $f \cdot F_i = 0$, for $i = 1, 2$, but $f \cdot F_3 = 6$.

Let $W = \langle F_1, F_2, F_3\rangle$ be the T-submodule of E generated by $\{F_1, F_2, F_3\}$. What is $I = \operatorname{ann}_T(W) = \{f \in T|\ f \cdot F_i = 0 \text{ for all } i\}$? Observe that if $\deg(f) \geq 4$ for $f \in T$, then $f \cdot F_i = 0$ for all i, i.e., $f \in I$. Also, $y_2^3 \cdot F_3 = 6 \neq 0$ implies that $y_2^3 \notin I$.
One can check that $I = \langle y_1^3, y_1^2y_2, y_1y_2^3, y_2^4\rangle$. Moreover W is an R-module, where $R = T/I$. [1]

By $\bar{y}_1$ and $\bar{y}_2$, we denote the respective images of y_1 and y_2 in R. Using $\mathbb{C}$-bases for R and W respectively, we can represent their R-module structures pictorially as follows:



It can be shown that $W \simeq \frac{Re_1 \oplus Re_2 \oplus Re_3}{\langle y_1e_3, y_2e_3 - y_1e_2, y_2^2e_2 - y_1e_1, y_2e_1\rangle}$ as an R-module.

Notice from the pictures that R is W turned 'upside down'. How are they related? It can be checked that $W = \operatorname{ann}_E(I)$, and the R-module W is a 'dual' module of R.

[1] Exercise: If M is an R-module, then it is also a module over $R/\operatorname{ann}_R(M)$.

Definition 2.2. *Let $T = \mathbb{C}[y_1, \dots, y_d]$, $E = \mathbb{C}[Y_1, \dots, Y_d]$, and I an ideal in T such that $R = T/I$ is Artinian. Let $W = \operatorname{ann}_E(I)$. Then*
i) *W is called a canonical module of R.*
ii) *R is said to be Gorenstein Artin if $R \simeq W$.*

Exercise. Let W be a finitely generated T-submodule of E. If $I = \operatorname{ann}_T(W)$, then $R = T/I$ is Artinian.

Observe that if R is Gorenstein Artin, then W is cyclic, i.e., there is an $F \in E$ such that $W = \langle F \rangle$, and hence $I = \operatorname{ann}_T(F)$.

Exercise. The converse is also true, i.e., if $I = \operatorname{ann}_T(F)$ for some $F \in E$, then $R = T/I$ is Gorenstein Artin.

3. Idealisation and the Corresponding Polynomial

Let the notation be as in Definition 2.2. Let $F_1, \cdots, F_r \in E$, $W = \langle F_1, \dots, F_r \rangle$, and $I = \operatorname{ann}_T(W)$. Then $R = T/I$ is Artinian. Let $E' = E[Z_1, \dots, Z_r]$, and $T' = T[z_1, \dots, z_r]$, where T' acts by partial differentiation on E', (i.e., $z_i \cdot Z_j = \delta_{ij}$), extending the action of T on E. Consider the polynomial $F = Z_1F_1 + \cdots + Z_rF_r \in E'$. Note that $z_i \cdot F = F_i$.

The question one asks is: To which Gorenstein Artin ring does this F correspond, and how is it related to R or W?

Theorem 3.1. *The polynomial F corresponds to the idealisation of W, $S = R \ltimes W$, i.e., $T'/\operatorname{ann}_{T'}(F) \simeq R \ltimes W$. In particular, $R \ltimes W$ is a Gorenstein Artin local ring.*

Let us illustrate this by the following example.

With notation as in Example 2.1, let $T' = \mathbb{C}[y_1, y_2, z_1, z_2, z_3]$, $E' = \mathbb{C}[Y_1, Y_2, Z_1, Z_2, Z_3]$, and $F = Z_1F_1 + Z_2F_2 + Z_3F_3$. Observe that $K = \langle z_1, z_2, z_3 \rangle^2 \subset \operatorname{ann}_{T'}(F)$, and $I = \langle y_1^3, y_2^2y_2, y_1y_2^3, y_2^4 \rangle \subset \operatorname{ann}_{T'}(F)$. Finally, $J = \langle y_1z_3, y_2z_3 - y_1z_2, y_2^2z_2 - y_1z_1, y_2z_1 \rangle \subset \operatorname{ann}_{T'}(F)$. It can be checked that $\operatorname{ann}_{T'}(F) = I + J + K$.

Let $S = R \ltimes W$. Since $W \simeq \frac{Re_1 \oplus Re_2 \oplus Re_3}{\langle y_1e_3, y_2e_3 - y_1e_2, y_2^2e_2 - y_1e_1, y_2e_1 \rangle}$ as an R-module, and $R \simeq \mathbb{C}[y_1, y_2]/I$, we get $S \simeq T'/(I + J + K) = T'/\operatorname{ann}_{T'}(F)$, illustrating Theorem 3.1. □

References

[1] H. Ananthnarayan, *Approximating Artinian rings by Gorenstein rings and three-standardness of the maximal ideal*, Ph.D. Thesis, University of Kansas (2009).
[2] N. Jacobson, *Basic Algebra I*, W. H. Freeman and Company, New York, 1985. xviii+499 pp.

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