RINGS WITH AT MOST TWO MAXIMAL IDEALS, DIRECT SUMS AND PRODUCTS

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1. Introduction and preliminary results

As in my previous preparatory talk, R will denote a ring with identity, not necessarily commutative, and M_R a unital right R-module.

We begin with the following theorem, which we had already presented in the previous preparatory talk.

Theorem 1. (Krull-Schmidt-Azumaya Theorem, 1950) *Let M be a module that is a direct sum of modules with local endomorphism rings. Then M is a direct sum of indecomposable modules in an essentially unique way in the following sense. If*

$$M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j,$$

where all the M_i's ($i \in I$) and all the N_j's ($j \in J$) are indecomposable modules, then there exists a bijection $\varphi\colon I \to J$ such that $M_i \cong N_{\varphi(i)}$ for every $i \in I$.

In general, for an arbitrary module M_R that is not a direct sum of modules with local endomorphism rings, the "essential uniqueness" of the statement of the previous theorem does not hold. Our initial aim in this talk is to describe direct-sum decompositions of an arbitrary module M_R as a direct sum $M_R = M_1 \oplus \cdots \oplus M_n$ of finitely many direct summands M_i. A number of behaviors is possible. For instance:

(1) For some modules, there is essential uniqueness of direct-sum decomposition into indecomposables, as in the case of modules of finite composition length (the classical Krull-Schmidt Theorem).

(2) There are modules M_R with a direct-sum decomposition into indecomposables, this decomposition is not unique in the sense of the Krull-Schmidt-Azumaya Theorem, but M_R has only finitely many direct-sum decompositions up to isomorphism. This happens, for instance, for torsion-free abelian groups $M_R = G_{\mathbb{Z}}$ of finite torsion-free rank.

(3) For some classes of modules, direct-sum decompositions into indecomposables are not unique, but they enjoy some kind of regularity, such as for the classes of modules we will see in Section **??**.

(4) But, in general, there is no direct-sum decomposition into indecomposables, and no form of uniqueness.

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The best algebraic way to describe direct-sum decompositions of a module M_R is making use of commutative monoids (semigroups with a binary operation that is associative, commutative and has an identity element). Let us quickly recall the main result in this direction. In this talk, all monoids S will be commutative and additive. A monoid S is said to be *reduced* if $s, t \in S$ and $s+t=0$ imply $s=t=0$.

We will also need some language of categories. Our categories will be mainly full subcategories of the category Mod-R of all right R-modules. Sometimes we will also identify classes of right R-modules with the corresponding full subcategory of Mod-R.

Let $\mathcal{C}$ be a category and $V(\mathcal{C})$ denote a *skeleton* of $\mathcal{C}$, that is, a class of representatives of the objects of $\mathcal{C}$ modulo isomorphism. For every object A in $\mathcal{C}$, there is a unique object $\langle A\rangle$ in $V(\mathcal{C})$ isomorphic to A. Thus there is a mapping $\mathrm{Ob}(\mathcal{C}) \to V(\mathcal{C})$, $A \mapsto \langle A\rangle$, that associates to every object A of $\mathcal{C}$ the unique object $\langle A\rangle$ in $V(\mathcal{C})$ isomorphic to A.

Assume that a product $A \times B$ exists in $\mathcal{C}$ for every pair A, B of objects of $\mathcal{C}$. Define an addition $+$ in $V(\mathcal{C})$ by $A + B := \langle A \times B\rangle$ for every $A, B \in V(\mathcal{C})$.

Lemma 2. *Let $\mathcal{C}$ be a category with a terminal object and in which a product $A \times B$ exists for every pair A, B of objects of $\mathcal{C}$. Then $V(\mathcal{C})$ is a large reduced commutative monoid.*

Conversely:

Theorem 3. [**?**, **?**] *Let k be a field and let M be a commutative reduced monoid. Then there exists a class $\mathcal{C}$ of finitely generated projective right modules over a right and left hereditary k-algebra R such that $M \cong V(\mathcal{C})$.*

2. Endomorphism rings with at most two maximal ideals and direct sums

Now we will examine the behavior of an important class of modules, that of uniserial modules. A module U_R is *uniserial* if the lattice $\mathcal{L}(U_R)$ of its submodules is linearly ordered under inclusion.

The endomorphism ring of a uniserial module has at most two maximal right (left) ideals:

Theorem 4. [**?**] *Let U_R be a uniserial module over a ring R, $E := \mathrm{End}(U_R)$ be its endomorphism ring, $I := \{\, f \in E \mid f$ is not injective$\,\}$ and $K := \{\, f \in E \mid f$ is not surjective$\,\}$. Then I and K are two two-sided completely prime ideals of E (that is, they satisfy the commutative definition of "prime"), and every proper right ideal of E and every proper left ideal of E is contained either in I or in K. Moreover,*

(a) *either E is a local ring with maximal ideal $I \cup K$, or*

(b) *E/I and E/K are division rings, and $E/J(E) \cong E/I \times E/K$.*

Thus the endomorphism ring of a uniserial module U_R has either one maximal ideal (that is, the endomorphism ring is local, case (a) in Theorem **??**) or two maximal ideals (case (b) in Theorem **??**). Let us see how the fact that the endomorphism ring can have two maximal ideals is reflected in direct sums of uniserial modules.

Two modules U and V are said to have

(1) *the same monogeny class*, denoted $[U]_m = [V]_m$, if there exist a monomorphism $U \to V$ and a monomorphism $V \to U$;
(2) *the same epigeny class*, denoted $[U]_e = [V]_e$, if there exist an epimorphism $U \to V$ and an epimorphism $V \to U$.

Theorem 5. (Weak Krull-Schmidt Theorem for uniserial modules [**?**]) *Let $U_1, \dots, U_n, V_1, \dots, V_t$ be $n+t$ non-zero uniserial right modules over a ring R. Then the direct sums $U_1 \oplus \cdots \oplus U_n$ and $V_1 \oplus \cdots \oplus V_t$ are isomorphic R-modules if and only if $n = t$ and there exist two permutations σ and τ of $\{1, 2, \dots, n\}$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i = 1, 2, \dots, n$.*

The behavior of uniserial modules described in Theorems **??** and **??** is enjoyed by other classes of modules, for instance by cyclically presented modules over a local ring [**?**]. A right module over a ring R is *cyclically presented* if it is isomorphic to R/aR for some element $a \in R$. For any ring R, the group of all invertible elements of R will be denoted by $U(R)$.

If R/aR and R/bR are cyclically presented modules over a local ring R, we say that R/aR and R/bR *have the same lower part*, and write $[R/aR]_l = [R/bR]_l$, if there exist $u, v \in U(R)$ and $r, s \in R$ with $au = rb$ and $bv = sa$. (It would be possible to prove that two cyclically presented modules over a local ring have the same lower part if and only if their Auslander-Bridger transposes have the same epigeny class [**?**].)

The endomorphism ring $\mathrm{End}_R(R/aR)$ of a non-zero cyclically presented module R/aR is isomorphic to E/aR, where $E := \{\, r \in R \mid ra \in aR \,\}$ is the *idealizer* of the principal right ideal aR. Let $J(R)$ denote the Jacobson radical of R.

Theorem 6. *Let a be a non-zero non-invertible element of an arbitrary local ring R, let E be the idealizer of aR, and let E/aR be the endomorphism ring of the cyclically presented right R-module R/aR. Set $I := \{\, r \in R \mid ra \in aJ(R) \,\}$ and $K := J(R) \cap E$. Then I and K are two two-sided completely prime ideals of E containing aR, the union $(I/aR) \cup (K/aR)$ is the set of all non-invertible elements of E/aR, and every proper right ideal of E/aR and every proper left ideal of E/aR is contained either in I/aR or in K/aR. Moreover, exactly one of the following two conditions holds:*

(a) *Either I and K are comparable (that is, $I \subseteq K$ or $K \subseteq I$), in which case E/aR is a local ring, or*

(b) *I and K are not comparable, and in this case E/I and E/K are division rings, $J(E/aR) = (I \cap K)/aR$, and $(E/aR)/J(E/aR)$ is canonically isomorphic to the direct product $E/I \times E/K$.*

Theorem **??** is the analogue of Theorem **??**. Theorem **??** is the analogue of Theorem **??**.

Theorem 7. (Weak Krull-Schmidt Theorem for cyclically presented modules over a local ring) *Let $a_1, \dots, a_n, b_1, \dots, b_t$ be $n+t$ non-invertible elements of a local ring R. Then the direct sums $R/a_1R \oplus \cdots \oplus R/a_nR$ and $R/b_1R \oplus \cdots \oplus R/b_tR$ are isomorphic right R-modules if and only if $n = t$ and there exist two permutations σ, τ of $\{1, 2, \dots, n\}$ such that $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ and $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ for every $i = 1, 2, \dots, n$.*

The Weak Krull-Schmidt Theorem for cyclically presented modules has an immediate consequence as far as equivalence of matrices is concerned. Recall that two

$m \times n$ matrices A and B with entries in a ring R are said to be *equivalent* matrices, denoted $A \sim B$, if there exist an $m \times m$ invertible matrix P and an $n \times n$ invertible matrix Q with entries in R (that is, matrices in $U(M_m(R))$ and $U(M_n(R))$, respectively) such that $B = PAQ$. We denote by $\operatorname{diag}(a_1, \dots, a_n)$ the $n \times n$ diagonal matrix whose (i, i) entry is a_i and whose other entries are zero.

If R is a *commutative* local ring and $a_1, \dots, a_n, b_1, \dots, b_n$ are elements of R, then $\operatorname{diag}(a_1, \dots, a_n) \sim \operatorname{diag}(b_1, \dots, b_n)$ if and only if there exists a permutation σ of $\{1, 2, \dots, n\}$ with a_i and $b_{\sigma(i)}$ associates for every $i = 1, 2, \dots, n$. Here $a, b \in R$ are *associates* if they generate the same principal ideal of R.

If the ring R is local, but non-necessarily commutative, we have the following result:

Proposition 8. *Let $a_1, \dots, a_n, b_1, \dots, b_n$ be elements of a local ring R. Then $\operatorname{diag}(a_1, \dots, a_n) \sim \operatorname{diag}(b_1, \dots, b_n)$ if and only if there exist two permutations σ, τ of $\{1, 2, \dots, n\}$ with*

$$[R/a_iR]_l = [R/b_{\sigma(i)}R]_l \quad \textit{and} \quad [R/a_iR]_e = [R/b_{\tau(i)}R]_e$$

for every $i = 1, 2, \dots, n$.

Another class of modules with a behavior similar to the behavior of uniserial modules and to the behavior of cyclically presented modules over local rings is the class of kernels of morphisms between indecomposable injective modules. We will now describe this class.

A standard technique of homological algebra allows us to extend a morphism between two modules to their injective resolutions, as follows.

Assume that E_0, E_1, E_0', E_1' are indecomposable injective right modules over a ring R, and that $\varphi\colon E_0 \to E_1, \varphi'\colon E_0' \to E_1'$ are two right R-module morphisms. A morphism $f\colon \ker \varphi \to \ker \varphi'$ extends to a morphism $f_0\colon E_0 \to E_0'$. Now f_0 induces a morphism $\widetilde{f_0}\colon E_0/\ker \varphi \to E_0'/\ker \varphi'$, which extends to a morphism $f_1\colon E_1 \to E_1'$. Thus we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \ker \varphi \arrow[r] \arrow[d, "f"] & E_0 \arrow[r, "\varphi"] \arrow[d, "f_0"] & E_1 \arrow[d, "f_1"] \\
0 \arrow[r] & \ker \varphi' \arrow[r] & E_0' \arrow[r, "\varphi'"'] & E_1'.
\end{tikzcd}$$

The morphisms f_0 and f_1 are not uniquely determined by f.

Theorem 9. *Let E_0 and E_1 be indecomposable injective right modules over a ring R, and let $\varphi\colon E_0 \to E_1$ be a non-zero non-injective morphism. Let $S := \operatorname{End}_R(\ker \varphi)$ denote the endomorphism ring of $\ker \varphi$. Set $I := \{\, f \in S \mid$ the endomorphism f of $\ker \varphi$ is not a monomorphism$\,\}$ and $K := \{\, f \in S \mid$ the endomorphism f_1 of E_1 is not a monomorphism$\,\} = \{\, f \in S \mid \ker \varphi \subset f_0^{-1}(\ker \varphi)\,\}$. Then I and K are two two-sided completely prime ideals of S, and every proper right ideal of S and every proper left ideal of S is contained either in I or in K. Moreover, exactly one of the following two conditions holds:*

(a) *Either I and K are comparable (that is, $I \subseteq K$ or $K \subseteq I$), in which case S is a local ring with maximal ideal $I \cup K$, or*

(b) *I and K are not comparable, and in this case S/I and S/K are division rings and $S/J(S) \cong S/I \times S/K$.*

For a right module A_R over a ring R, let $E(A_R)$ denote the injective envelope of A_R. We say that two modules A_R and B_R *have the same upper part*, and write $[A_R]_u = [B_R]_u$, if there exist a homomorphism $\varphi\colon E(A_R) \to E(B_R)$ and a homomorphism $\psi\colon E(B_R) \to E(A_R)$ such that $\varphi^{-1}(B_R) = A_R$ and $\psi^{-1}(A_R) = B_R$.

Theorem 10. (Weak Krull-Schmidt Theorem [**?**]) *Let $\varphi_i\colon E_{i,0} \to E_{i,1}$ $(i = 1, 2, \dots, n)$ and $\varphi'_j\colon E'_{j,0} \to E'_{j,1}$ $(j = 1, 2, \dots, t)$ be $n + t$ non-injective morphisms between indecomposable injective right modules $E_{i,0}, E_{i,1}, E'_{j,0}, E'_{j,1}$ over an arbitrary ring R. Then the direct sums $\oplus_{i=0}^{n} \ker \varphi_i$ and $\oplus_{j=0}^{t} \ker \varphi'_j$ are isomorphic R-modules if and only if $n = t$ and there exist two permutations σ, τ of $\{1, 2, \dots, n\}$ such that $[\ker \varphi_i]_m = [\ker \varphi'_{\sigma(i)}]_m$ and $[\ker \varphi_i]_u = [\ker \varphi'_{\tau(i)}]_u$ for every $i = 1, 2, \dots, n$.*

There are some further classes of modules with the same behavior:

(1) Couniformly presented modules [**?**].

(2) Biuniform modules (modules of Goldie dimension one and dual Goldie dimension one) [**?**].

(3) Another class of modules that can be described via two similar invariants is that of Auslander-Bridger modules. For Auslander-Bridger modules, the two invariants are epi-isomorphism and lower-isomorphism [**?**].

The general categorical pattern for the results in this section is the following.

Let $\mathcal{C}$ be a full subcategory of the category Mod-R for some ring R and assume that every object of $\mathcal{C}$ is an indecomposable right R-module. Define a *completely prime ideal* $\mathcal{P}$ of $\mathcal{C}$ as an assignment of a subgroup $\mathcal{P}(A, B)$ of the additive abelian group $\mathrm{Hom}_R(A, B)$ to every pair (A, B) of objects of $\mathcal{C}$ with the following two properties: (1) for every $A, B, C \in \mathrm{Ob}(\mathcal{C})$, every $f\colon A \to B$ and every $g\colon B \to C$, one has that $gf \in \mathcal{P}(A, C)$ if and only if either $f \in \mathcal{P}(A, B)$ or $g \in \mathcal{P}(B, C)$; (2) $\mathcal{P}(A, A)$ is a proper subgroup of $\mathrm{Hom}_R(A, A)$ for every object $A \in \mathrm{Ob}(\mathcal{C})$.

Let $\mathcal{P}$ be a completely prime ideal of $\mathcal{C}$. If A, B are objects of $\mathcal{C}$, we say that A and B *have the same $\mathcal{P}$ class*, and write $[A]_{\mathcal{P}} = [B]_{\mathcal{P}}$, if $\mathcal{P}(A, B) \neq \mathrm{Hom}_R(A, B)$ and $\mathcal{P}(B, A) \neq \mathrm{Hom}_R(B, A)$.

Theorem 11. [**?**] *Let $\mathcal{C}$ be a full subcategory of* Mod-R *and $\mathcal{P}, \mathcal{Q}$ be two completely prime ideals of $\mathcal{C}$. Assume that all objects of $\mathcal{C}$ are indecomposable right R-modules and that, for every $A \in \mathrm{Ob}(\mathcal{C})$, $f\colon A \to A$ is an automorphism of A if and only if $f \notin \mathcal{P}(A, A) \cup \mathcal{Q}(A, A)$. Then, for every $A_1, \dots, A_n, B_1, \dots, B_t \in \mathrm{Ob}(\mathcal{C})$, the modules $A_1 \oplus \dots \oplus A_n$ and $B_1 \oplus \dots \oplus B_t$ are isomorphic if and only if $n = t$ and there exist two permutations σ, τ of $\{1, 2, \dots, n\}$ such that $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$ and $[A_i]_{\mathcal{Q}} = [B_{\tau(i)}]_{\mathcal{Q}}$ for all $i = 1, \dots, n$.*

For the classes $\mathcal{C}$ of modules described until now, the fact that the weak form of the Krull-Schmidt Theorem holds can be described saying that the corresponding monoid $V(\mathcal{C})$ is a subdirect product of two free monoids.

3. Direct products of infinite families of modules

Until now we have considered finite direct sums, of uniserial modules for instance. There is a corresponding theory for infinite direct sums of uniserial modules ([**?**], [**?**]), but we won't present it here. Instead, we will consider what occurs for infinite direct products of uniserial modules. (Of course, finite direct sums and finite direct products coincide.)

Theorem 12. [?] *Let $\{ U_i \mid i \in I \}$ and $\{ V_j \mid j \in J \}$ be two families of uniserial modules over an arbitrary ring R. Assume that there exist two bijections $\sigma, \tau \colon I \to J$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I$. Then $\prod_{i\in I} U_i \cong \prod_{j\in J} V_j$.*

The general pattern is the following. We say that a full subcategory $\mathcal{C}$ of Mod-R *satisfies Condition (DSP)* (direct summand property) if whenever A, B, C, D are right R-modules with $A \oplus B \cong C \oplus D$ and $A, B, C \in \mathrm{Ob}(\mathcal{C})$, then also $D \in \mathrm{Ob}(\mathcal{C})$.

Theorem 13. [?] *Let $\mathcal{C}$ be a full subcategory of* Mod-R *in which all objects are indecomposable right R-modules and let $\mathcal{P}, \mathcal{Q}$ be two completely prime ideals of $\mathcal{C}$ with the property that, for every $A \in \mathrm{Ob}(\mathcal{C})$, an endomorphism $f \colon A \to A$ is an automorphism if and only if $f \notin \mathcal{P}(A, A) \cup \mathcal{Q}(A, A)$. Assume that $\mathcal{C}$ satisfies Condition (DSP). Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of objects of $\mathcal{C}$. Assume that there exist two bijections $\sigma, \tau \colon I \to J$ such that $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$ and $[A_i]_{\mathcal{Q}} = [B_{\tau(i)}]_{\mathcal{Q}}$ for every $i \in I$. Then the R-modules $\prod_{i\in I} A_i$ and $\prod_{j\in J} B_j$ are isomorphic.*

For instance, for cyclically presented modules over a local ring, we find that:

Theorem 14. *Let R be a local ring and $\{ U_i \mid i \in I \}$ and $\{ V_j \mid j \in J \}$ be two families of cyclically presented right R-modules. Suppose that there exist two bijections $\sigma, \tau \colon I \to J$ such that $[U_i]_l = [V_{\sigma(i)}]_l$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I$. Then $\prod_{i\in I} U_i \cong \prod_{j\in J} V_j$.*

Similarly, for kernels of morphisms between indecomposable injective modules, we get that:

Theorem 15. *Let R be a ring and $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of right R-modules that are all kernels of non-injective morphisms between indecomposable injective modules. Suppose that there exist bijections $\sigma, \tau \colon I \to J$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ and $[A_i]_u = [B_{\tau(i)}]_u$ for every $i \in I$. Then $\prod_{i\in I} A_i \cong \prod_{j\in J} B_j$.*

And now we will try to reverse the results in this Section. Is it possible to do it? For example, is it possible to invert the implication in Theorem **??**? That is, does a direct product of uniserial modules completely determine the monogeny classes and the epigeny classes of the factors? The answer is "no", it isn't possible, as the following three examples show.

(1) Let R be the localization of the ring $\mathbb{Z}$ of integers at a maximal ideal (p), and consider the direct product $\mathbb{Q} \oplus (\mathbb{Z}(p^\infty))^{\mathbb{N}^*} \cong (\mathbb{Z}(p^\infty))^{\mathbb{N}^*}$. In these two isomorphic direct products, all the factors are uniserial R-modules with a local endomorphism ring, but there are no bijections preserving the monogeny classes and the epigeny classes in the two direct products.

(2) Let R be the ring $\mathbb{Z}$ of integers and $\mathcal{C}$ be the full subcategory of Mod-R whose objects are all injective indecomposable R-modules. If A and B are objects of $\mathcal{C}$, let $\mathcal{P}(A, B)$ be the group of all morphisms $A \to B$ that are not automorphisms, so that $\mathcal{P}$ is a completely prime ideal of $\mathcal{C}$, $\mathbb{Q} \oplus \prod_p \mathbb{Z}(p^\infty) \cong \prod_p \mathbb{Z}(p^\infty)$, but there does not exist a bijection σ preserving the $\mathcal{P}$ classes.

(3) Let p be a prime number, and let $\widehat{\mathbb{Z}_p}$ be the ring of p-adic integers, so that $\mathbb{Z}/p^n\mathbb{Z}$ is a module over $\widehat{\mathbb{Z}_p}$ for every integer $n \geq 1$. Then $\widehat{\mathbb{Z}_p} \oplus \prod_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z} \cong \prod_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z}$. In these direct products, all the factors $\widehat{\mathbb{Z}_p}$ and $\mathbb{Z}/p^n\mathbb{Z}$ $(n \geq 1)$ are

pair-wise non-isomorphic uniserial $\widehat{\mathbb{Z}_p}$-modules, have distinct monogeny classes and distinct epigeny classes, and therefore there cannot be bijections σ and τ preserving the monogeny and the epigeny classes in the two direct-product decompositions.

Thus it is not possible to invert the results of this Section in general; for instance it is not possible to invert Theorem **??**. But for a special class of modules, that of slender modules, it is possible. Let us recall what slender modules are. (A good reference for slender modules is [**?**].) Let R be a ring. Let $R^\omega := \prod_{n<\omega} e_n R$ be a right R-module that is the direct product of countably many copies of the right R-module R_R, where e_n is the element of R^ω with support $\{n\}$ and equal to 1 in n. A right R-module M_R is *slender* if, for every homomorphism $f\colon R^\omega \to M$ there exists $n_0 < \omega$ such that $f(e_n) = 0$ for all $n \geq n_0$.

Theorem 16. *A module M_R is slender if and only if, for every countable family $\{ P_n \mid n \geq 0 \}$ of right R-modules and any homomorphism $f\colon \prod_{n\geq 0} P_n \to M_R$, there exists $m \geq 0$ such that $f(\prod_{n\geq m} P_n) = 0$.*

Slender modules are naturally related to a very large class of cardinals, the class of cardinals that are non-measurable. A cardinal α is *measurable* if it is an uncountable cardinal with an α-complete, non-principal ultrafilter. An ultrafilter is *α-complete* if the intersection of any *strictly less than* α-many sets in the ultrafilter is also in the ultrafilter.

If a cardinal is not measurable, then neither are all smaller cardinals. Thus if there exists a measurable cardinal, then there is a smallest one and all larger cardinals are measurable.

It is not known whether ZFC $\Rightarrow \exists$ a measurable cardinal.

Here is why non-measurable cardinals appear naturally in the study of slender modules:

Theorem 17. *If M_R is slender and $\{ P_i \mid i \in I \}$ is a family of right R-modules with $|I|$ non-measurable, then $\operatorname{Hom}(\prod_{i\in I} P_i, M_R) \cong \bigoplus_{i\in I} \operatorname{Hom}(P_i, M_R)$.*

Clearly, every submodule of a slender module is a slender module. Also:

Theorem 18. *A $\mathbb{Z}$-module is slender if and only if it does not contains a copy of $\mathbb{Q}$, $\mathbb{Z}^\omega$, $\mathbb{Z}/p\mathbb{Z}$ or $\widehat{\mathbb{Z}_p}$ for any prime p.*

The general result we have in this setting is the following rather technical result.

Theorem 19. [**?**] *Let $\mathcal{C}$ be a full subcategory of* Mod-R *in which all objects are indecomposable slender right R-modules and let $\mathcal{P}$, $\mathcal{Q}$ be a pair of completely prime ideals of $\mathcal{C}$ with the property that, for every $A \in \mathrm{Ob}(\mathcal{C})$, $f\colon A \to A$ is an automorphism if and only if $f \notin \mathcal{P}(A,A) \cup \mathcal{Q}(A,A)$. Assume that $\mathcal{C}$ satisfies Condition (DSP). Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of objects of $\mathcal{C}$ with $|I|$ and $|J|$ non-measurable. Assume that:*

(a) *In both families, there are at most countably many modules in each $\mathcal{P}$ class.*
(b) *In both families, there are at most countably many modules in each $\mathcal{Q}$ class.*
(c) *The R-modules $\prod_{i\in I} A_i$ and $\prod_{j\in J} B_j$ are isomorphic.*

Then there exist two bijections $\sigma, \tau\colon I \to J$ such that $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$ and $[A_i]_{\mathcal{Q}} = [B_{\tau(i)}]_{\mathcal{Q}}$ for every $i \in I$.

We conclude with three results that follow from Theorem **??**.

Corollary 20. *Let $\mathcal{C}$ be a full subcategory of* Mod-R *in which all objects are indecomposable slender right R-modules and let $\mathcal{P}, \mathcal{Q}$ be a pair of completely prime ideals of $\mathcal{C}$ with the property that, for every $A \in \mathrm{Ob}(\mathcal{C})$, $f\colon A \to A$ is an automorphism if and only if $f \notin \mathcal{P}(A, A) \cup \mathcal{Q}(A, A)$. Assume that $\mathcal{C}$ satisfies Condition (DSP). Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two countable families of objects of $\mathcal{C}$. Assume that $\prod_{i\in I} A_i \cong \prod_{j\in J} B_j$. Then there exist two bijections $\sigma, \tau\colon I \to J$ such that $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$ and $[A_i]_{\mathcal{Q}} = [B_{\tau(i)}]_{\mathcal{Q}}$ for every $i \in I$.*

Theorem 21. *Let $\mathcal{C}$ be a full subcategory of* Mod-R *in which all objects are slender right R-modules and let $\mathcal{P}$ be a completely prime ideal of $\mathcal{C}$. Let $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$ be two families of objects of $\mathcal{C}$ with $|I|$ and $|J|$ non-measurable. Assume that:*

(a) *For every object A of $\mathcal{C}$, $\mathcal{P}(A, A)$ is a maximal right ideal of* $\mathrm{End}_R(A)$.

(b) *There are at most countably many modules in each $\mathcal{P}$ class in both families $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$.*

(c) *The R-modules $\prod_{i\in I} A_i$ and $\prod_{j\in J} B_j$ are isomorphic.*

Then there is a bijection $\sigma\colon I \to J$ such that $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$ for every $i \in I$.

Corollary 22. [**?**] *Let R be a ring and $\{ A_i \mid i \in I \}$ be a family of slender right R-modules with local endomorphism rings. Let $\{ B_j \mid j \in J \}$ be a family of indecomposable slender right R-modules. Assume that:*

(a) *$|I|$ and $|J|$ are non-measurable cardinals.*

(b) *There are at most countably many mutually isomorphic modules in each of the two families $\{ A_i \mid i \in I \}$ and $\{ B_j \mid j \in J \}$.*

(c) *The R-modules $\prod_{i\in I} A_i$ and $\prod_{j\in J} B_j$ are isomorphic.*

Then there exists a bijection $\sigma\colon I \to J$ such that $A_i \cong B_{\sigma(i)}$ for every $i \in I$.

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