NON-COMMUTATIVE LOCAL RINGS

ALBERTO FACCHINI
UNIVERSITÀ DI PADOVA, ITALY
PREPARATORY TALK
21 APRIL 2015

In this talk, we deal with rings R with an identity, *not necessarily commutative.* A *right ideal* I of R is a subgroup of the additive group of R such that $xr \in I$ for every $x \in I$ and every $r \in R$. That is,

(i) $I \subseteq R$;
(ii) $I \neq \emptyset$;
(iii) $x - y \in I$ for every $x, y \in I$;
(iv) $xr \in I$ for every $x \in I, r \in R$.

We will denote "I is a right ideal of R" by $I \leq R_R$. A *left ideal* I of R is a subgroup of the additive group of R such that $rx \in I$ for every $r \in R$ and every $x \in I$. We will denote "I is a left ideal of R" by $I \leq {}_RR$. A *two-sided ideal* of R (or, simply, an *ideal* of R) is a subset I of R that is both a right ideal of R and a left ideal of R. In this case, we will write $I \trianglelefteq R$. A right ideal, left ideal or two-sided ideal of R is *proper* if it is different from R.

For any module M_R, the *partially ordered set of all submodules of* M_R will be denoted by $\mathcal{L}(M_R)$. Its elements are the submodules of M_R. The partial order is set inclusion. The partially ordered set $(\mathcal{L}(M_R), \subseteq)$ is a lattice where $N \vee N' = N + N'$ and $N \wedge N' = N \cap N'$ for every $N, N' \in \mathcal{L}(M_R)$.

A submodule of M_R is a *maximal submodule* if it is a maximal element of the partially ordered set $\mathcal{L}(M_R) \setminus \{M_R\}$. Thus a submodule N of M_R is maximal if $N < M_R$ and, for every submodule P of M_R with $N \leq P$, either $P = N$ or $P = M_R$.

Proposition 1. *Every non-zero finitely generated module has at least one maximal submodule. More generally, every proper submodule of a finitely generated module* M_R *is contained in a maximal submodule of* M_R.

An element r of a ring R is said to be:

(i) *right invertible* if there exists $s \in R$ such that $rs = 1_R$;
(i) *left invertible* if there exists $s \in R$ such that $sr = 1_R$;
(iii) *invertible* if it is both right invertible and left invertible.

Here is an exercise for the reader. Let k be a field and V, W be vector spaces over k. Recall that a mapping $f\colon V \to W$ is called *k-linear,* or a *linear transformation,* if $f(v + v) = f(v) + f(v')$ and $f(\lambda v) = \lambda f(v)$ for every $v, v' \in V$ and every $\lambda \in k$. Assume $V \neq 0$. Let $\mathrm{End}_k(V)$ be the set of all k-linear mappings $f\colon V \to V$.

Date: September 15, 2016.

Define two operations on $\mathrm{End}_k(V)$ setting

$$(f+g)(v) := f(v) + g(v)$$
$$(fg)(v) := f(g(v))$$

for every $f, g \in \mathrm{End}_k(V)$ and every $v \in V$. Then $\mathrm{End}_k(V)$ turns out to be a ring in which the multiplication is the composition of linear mappings, the identity $1_{\mathrm{End}_k(V)}$ is the identity mapping $V \to V$, $v \mapsto v$ for every $v \in V$, and the zero $0 = 0_{\mathrm{End}_k(V)}$ is the zero mapping, that is, the constant mapping $V \to V$, $v \mapsto 0_V$ for every $v \in V$. The ring $\mathrm{End}_k(V)$ is called the *endomorphism ring* of the vector space k.

(1) Show that the following conditions are equivalent for an element $f \in \mathrm{End}_k(V)$:

(i) f is an injective mapping;
(ii) f is left invertible in the ring $\mathrm{End}_k(V)$.

(2) Show that the following conditions are equivalent for an element $f \in \mathrm{End}_k(V)$:

(i) f is a surjective mapping;
(ii) f is right invertible in the ring $\mathrm{End}_k(V)$.

(3) Show that an element $f \in \mathrm{End}_k(V)$ is invertible if and only if it is a bijection.

(4) Give examples of elements in $\mathrm{End}_k(V)$ that have different *left inverses* (but no right inverses), and examples of elements in $\mathrm{End}_k(V)$ that have different right inverses (but no left inverses).

A ring R is a *division ring* if every non-zero element of R is invertible.

Lemma 2. *The following conditions are equivalent for a ring R:*

(i) *R is a division ring.*
(ii) *Every non-zero element of R is right invertible.*
(iii) *Every non-zero element of R is left invertible.*
(iv) *The only right ideals of R are $\{0_R\}$ and R.*
(v) *The only left ideals of R are $\{0_R\}$ and R.*

The *Jacobson radical* $J(R)$ of R is the intersection of all maximal right ideals of R. It can be proved that the intersection of all maximal left ideals of R coincides with the intersection of all maximal right ideals of R.

Proposition 3. *The following conditions are equivalent for a ring R:*

(i) *The ring R has a unique maximal right ideal.*
(ii) *The Jacobson radical $J(R)$ is a maximal right ideal.*
(iii) *The sum of two elements of R that are not right invertible is not right invertible.*
(iv) $J(R) = \{\, r \in R \mid rR \neq R \,\}$.
(v) *$R/J(R)$ is a division ring.*
(vi) $J(R) = \{\, r \in R \mid r$ *is not invertible in* $R \,\}$.
(vii) *The sum of two non-invertible elements of R is non-invertible.*
(viii) *For every $r \in R$, either r is invertible or $1 - r$ is invertible.*

The rings that satisfy the equivalent conditions of Proposition **??** are called *local rings*. For instance, division rings are local rings.

A module M_R is *indecomposable* if $M_R \neq 0$ and whenever $M_R = A \oplus B$, with A, B submodules of M_R, then either $A = 0$ or $B = 0$. For example, simple modules are indecomposable. The $\mathbb{Z}$-module $\mathbb{Z}$ is not simple, but it is indecomposable, because the intersection of any two non-zero submodules of $\mathbb{Z}$ is non-zero.

Proposition 4. *Let M_R be a right module over an arbitrary ring R and assume that* $\mathrm{End}(M_R)$ *is a local ring. Then the R-module M_R is indecomposable.*

Theorem 5. (Krull-Schmidt-Azumaya Theorem, 1950) *Let M be a module that is a direct sum of modules with local endomorphism rings. Then M is a direct sum of indecomposable modules in an essentially unique way in the following sense. If*

$$M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j,$$

where all the submodules M_i $(i \in I)$ and all the submodules N_j $(j \in J)$ are indecomposable modules, then there exists a bijection $\varphi\colon I \to J$ such that $M_i \cong N_{\varphi(i)}$ for every $i \in I$.

We conclude by giving the two main examples of classes of modules whose endomorphism ring is local. We begin with modules of finite composition length. Indecomposable modules of finite composition length turn out to have local endomorphism rings.

A module M_R is said to be a module of *finite composiition length* if there exists a finite chain $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M_R$ of submodules of M_R with M_{i-1} a maximal submodule of M_i for every $i = 1, 2, \ldots, n$.

Here are some examples of modules of finite composition length.

(1) Simple modules M_R have the trivial composition series $0 = M_0 \subseteq M_1 = M_R$, hence simple modules are of finite composition length.

(2) For a fixed prime p, the Prüfer group $\mathbb{Z}(p^\infty)$ has no composition series. To see it, recall that the partially ordered set $\mathcal{L}(\mathbb{Z}(p^\infty))$ is isomorphic to the well ordered set $\{0, 1, 2, 3, \ldots, +\infty\}$ with its natural order, so that the last non-zero factor of any series of $\mathbb{Z}(p^\infty)$ cannot be simple.

(3) A $\mathbb{Z}$-module has a composition series if and only if it is finite.

Proposition 6. *Let R be a ring. A module M_R is of finite composition length if and only if M_R is both artinian and noetherian.*

As we have already said, the endomorphism ring $\mathrm{End}(M_R)$ of an indecomposable module M_R of finite composition length is local.

Another class of modules whose endomorphism rings are local is the class of indecomposable injective modules. Let us recall what injective modules are.

Proposition 7. *The following conditions are equivalent for an R-module E_R:*

(i) *The functor* $\mathrm{Hom}(-, E_R)\colon \text{Mod-}R \to \mathrm{Ab}$ *is exact, that is, for every exact sequence $0 \to M'_R \to M_R \to M''_R \to 0$ of right R-modules, the sequence of abelian groups* $0 \to \mathrm{Hom}(M''_R, E_R) \to \mathrm{Hom}(M_R, E_R) \to \mathrm{Hom}(M'_R, E_R) \to 0$ *is exact.*

(ii) *For every monomorphism $M'_R \to M_R$ of right R-modules, the homomorphism of abelian groups*

$$\mathrm{Hom}(M_R, E_R) \to \mathrm{Hom}(M'_R, E_R)$$

is an epimorphism.

(iii) *For every submodule M'_R of a right R-module M_R, every morphism $M'_R \to E_R$ can be extended to a morphism $M_R \to E_R$.*

(iv) *For every monomorphism $f\colon M'_R \to M_R$ and every homomorphism $g\colon M'_R \to E_R$, there exists a morphism $h\colon M_R \to E_R$ with $h \circ f = g$.*

A module E_R is *injective* if it satisfies the equivalent conditions of Proposition **??**.

Condition (iv) is described by the following commutative diagram, in which the row is exact:

$$\begin{array}{ccccc} 0 & \longrightarrow & M'_R & \xrightarrow{f} & M_R \\ & & & \searrow^{g} & \downarrow{\scriptstyle h} \\ & & & & E_R \end{array}$$

Proposition 8. *The endomorphism ring* $\mathrm{End}(E_R)$ *of an indecomposable injective module* E_R *is local.*

What we have presented in this lecture can be found in any text book of non-commutative rings and modules, for instance in F. W. Anderson and K. R. Fuller, “Rings and categories of modules”, Second edition, GTM **13**, Springer-Verlag, New York (1992).